TWISTED WESS-ZUMINO-WITTEN MODELS ON ELLIPTIC CURVES

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Abstract. Investigated is a variant of the Wess-Zumino-Witten model called a twisted WZW model, which is associated to a certain Lie group bundle on a family of elliptic curves. The Lie group bundle is a non-trivial bundle with flat connection and related to the classical elliptic r-matrix. (The usual (non-twisted) WZW model is associated to a trivial group bundle with trivial connection on a family of compact Riemann surfaces and a family of its principal bundles.) The twisted WZW model on a fixed elliptic curve at the critical level describes the XYZ Gaudin model. The elliptic Knizhnik-Zamolodchikov equations associated to the classical elliptic r-matrix appear as flat connections on the sheaves of conformal blocks in the twisted WZW model.

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0. Introduction

In this paper, we deal with a variant of the (chiral) Wess-Zumino-Witten model (WZW model, for short) on elliptic curves, which shall be called a twisted WZW model. The usual (non-twisted) WZW model on a compact Riemann surface X gives rise to the sheaves of vector spaces (of conformal blocks and of conformal coinvariants) on any family (or the moduli stack) of principal G-bundles, where G is a semisimple complex algebraic group. Note that the notion of principal G-bundles is equivalent to that of G^{nt}-torsors, where G^{nt} denotes the trivial group bundle $G \times X$ on X. (The symbol $(\cdot)^{\mathrm{nt}}$ stands for "non-twisted".) This suggests that there exists a model associated to a non-trivial group bundle G^{tw} with a flat connection on X, which gives sheaves of conformal blocks and conformal coinvariants on a family of G^{tw}-torsors. (The symbol $(\cdot)^{\mathrm{tw}}$ stands for "twisted".) We call such a model a *twisted WZW model* associated to G^{tw}.

The aim of this work is not to establish a general theory of the twisted WZW models but to describe certain interesting examples of the twisted WZW models related to the elliptic classical r-matrices ([BelD]). In this introduction, we explain our motivations and clarify the relationship between the twisted WZW models and various problems in mathematics and physics.

One of the motivations is the viewpoint of representation theory where the WZW model is formulated as an analogue of a theory of automorphic forms due to Langlands. We list corresponding ingredients of both theories in Table 1. Notations shall be explained in the main part of this paper.

There is a theory of automorphic forms for arbitrary (possibly non-split) reductive groups over a global field as well as over a local field. But so far only the WZW model associated to the trivial group bundle has been considered and the counterpart of the non-split reductive group over a global field has been absent. The twisted WZW models fills this blank.

The second motivation comes from the geometric Langlands program over $\mathbb{C}$ and its relation with quantum integrable systems. A geometric analogue of the Langlands correspondence over $\mathbb{C}$ is described by using the WZW model at the critical level, where the centers of the completed enveloping algebras of affine Lie algebras are sufficiently large so that we can consider analogues of the infinitesimal characters of finite-dimensional semisimple Lie algebras ([Hay], [GW]). For introduction to the original Langlands program we refer to [Bo] and [Ge]. For a general formulation of the geometric Langlands correspondence over $\mathbb{C}$ related to the non-twisted WZW models at the critical level, see [Bei] and [BeiD] and for the analogue of the local Langlands correspondence to affine Lie algebras at the critical level, see [FF3] and [Fr1]. The twisted WZW model at the critical level shall give a geometric analogue of the Langlands correspondence for a non-split reductive group over a global field.

To study this model at the critical level is important not only in this context of the geometric Langlands program but also in the theory of the quantum integrable

Table 1. Analogy between automorphic forms and conformal blocks

Theory of automorphic forms	Theory of the WZW models
a global field, i.e., a number field or the function field of an algebraic curve over a finite field	the function field of a compact Riemann surface X
a local field	$\mathbb{C}((\xi))$, a field of formal Laurent series
a reductive group over the global field	a semisimple group bundle on X with flat connection or the associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$
a non-split reductive group over the global field	a semisimple group bundle with flat connection on X which is not locally trivial under the Zariski topology
the adèle group associated to the reductive group	the affine Lie algebra $(\mathfrak{g}^{\oplus L})^\wedge = \bigoplus_{i=1}^L \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}\hat{k}$
the principal adèle subgroup of the adèle group	the subalgebra $\mathfrak{g}_X^D = H^0(X, \mathfrak{g}^{\mathrm{tw}}(*D))$ of $(\mathfrak{g}^{\oplus L})^\wedge$
a unitary representation of the adèle group	a representation M of $(\mathfrak{g}^{\oplus L})^\wedge$ or its algebraic dual M^*
the space of automorphic forms in the representation space, i.e., the invariant subspace of the representation space with respect to the principal adèle subgroup	the space $\mathrm{CB}(M)$ of conformal blocks, i.e., the invariant subspace of M^* with respect to $\mathfrak{g}_X^D$

spin chains. B. Feigin, E. Frenkel and N. Reshetikhin found in [FFR] that the non-twisted WZW model at the critical level on the Riemann sphere is closely related to a spin chain model called the Gaudin model. See also [Fr2]. Its Hamiltonian is described as an insertion of a singular vector of the vacuum representation at a point and the diagonalization problem turns out to be equivalent to a description of a certain space of conformal blocks. This "Gaudin" model is, however, merely a special case of the model introduced by M. Gaudin [Ga1], [Ga2], [Ga3] as a quasi-classical limit of the XYZ spin chain model. Let us call this general model the XYZ Gaudin model, following [ST1] where the diagonalization problem of this model was studied by the algebraic Bethe Ansatz. In order to extend the results of [FFR], we need the twisted WZW model on a elliptic curve at the critical level as we shall see in §2.

We remark that the non-twisted WZW models at the critical level on an elliptic curve is related to quantum integrable systems on root systems. In fact those systems defined by the trigonometric (dynamical) r-matrices are described by the non-twisted WZW model on a degenerated elliptic curve with only one ordinary double point. The non-twisted WZW model at the critical level on an elliptic curve leads to a system called the Gaudin-Calogero model ([ER], [N]) which was defined as Hitchin's classical integrable system ([Hi]) on the moduli space of semistable principal bundles on an elliptic curve.[1] The reason why root systems appear in the

[1]This relation of the non-twisted WZW model and the Gaudin-Calogero model is due to B. Enriquez and A. Stoyanovsky. TT thanks Enriquez for communicating their unpublished result.

non-twisted WZW models is explained as follows. Let G be a complex semisimple group and T its maximal torus. Let a and b denote generators of the fundamental group $\pi_1(X)$ of an elliptic curve X. Then, for $g \in T$, the homomorphism from $\pi_1(X)$ into G sending a and b to 1 and g respectively induces a semistable principal G-bundle on X. This defines the covering by T of the moduli space of semistable G-bundles on X. Furthermore the universal covering of T is identified with its Lie algebra, on which the root system structure exists. Namely, the root system appears as a covering space of the moduli space of semistable principal G-bundles on X.

The third motivation is a geometric interpretation of Etingof's elliptic KZ equations. As is well-known, the Knizhnik-Zamolodchikov equation is a system of differential equations satisfied by matrix elements of products of vertex operators ([KZ], [TK]) and is a flat connection over the family of pointed Riemann spheres. Similarly from the non-twisted WZW model over elliptic curves arises the elliptic Knizhnik-Zamolodchikov-Bernard equations (KZB equations, for short), which Bernard found in [Be1] by computing traces of products of vertex operators twisted by $g \in G$. The interpretation of the elliptic KZB equations as flat connections on sheaves of conformal blocks, which are defined without use of the traces, was found in [FW]. Using the same idea as [Be1], Etingof computed in [E] a twisted trace of a product of vertex operators and found that it obeys linear differential equations of KZ type defined by the elliptic classical r-matrices. We call these equations the elliptic KZ equations. In the present paper it is shown that the elliptic KZ equations also has an interpretation as flat connections on sheaves of conformal blocks.

Let us explain now the contents of this paper. In §1, we give a definition of the conformal coinvariants and the conformal blocks of the twisted WZW model on an elliptic curve. The definition of the non-trivial group bundle G^{tw} (1.3) and the associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ (1.4) is given in §1.1 and their fundamental properties are studied. This bundle $\mathfrak{g}^{\mathrm{tw}}$ was used by I. Cherednik [C] for a algebro-geometric interpretation of classical elliptic r-matrices. An important point is that the cohomology groups of $\mathfrak{g}^{\mathrm{tw}}$ vanish in all degrees. Since the 1-cohomology $H^1(X, \mathfrak{g}^{\mathrm{tw}})$ can be canonically identified with the tangent space of the moduli space of G^{tw}-torsors at the equivalence class consisting of trivial ones, the trivial G^{tw}-torsor can not be deformed. Thus non-trivial G^{tw}-torsors do not appear in the twisted WZW model associated to G^{tw}. Conformal coinvariants and conformal blocks of this model are defined in §1.2. We introduce correlation functions of current and the energy-momentum tensor in §1.3, following mostly [TUY]. An action of the Virasoro algebra on the conformal coinvariants and the conformal blocks is defined in §1.4.

This model at the critical level for $G = SL_2(\mathbb{C})$ describes the XYZ Gaudin model and the case for $G = SL_N(\mathbb{C})$ is related to the higher rank generalizations of the XYZ Gaudin model, as shown in §2.

Away from the critical level, we can define a connection on the sheaves of conformal coinvariants and blocks over the family of pointed elliptic curves. §3 and §4 are a sheaf version of §1 over a family of pointed elliptic curves. By extending the tangent sheaf of the base space of the family (4.3) and constructing its action on the sheaves of conformal coinvariants and conformal blocks, we can introduce the D-module structure on them in §5, which naturally implies the existence of flat connections. The explicit formulae in §5.2 show that our connections are identical with Etingof's elliptic KZ equations. This connection has modular invariance which

Etingof proved by his explicit expressions of the equations. We give a geometric proof of this fact in §5.3.

Useful properties of theta functions are listed in Appendix A. An algebro-geometric meaning of the extension (4.3) is explained in Appendix B. Higher-genus generalization of the theory is discussed in Appendix C.

1. Spaces of conformal coinvariants and conformal blocks

In this section we define the space of conformal coinvariants and conformal blocks associated to a twisted Group bundle.

1.1. **Group bundles and their associated Lie algebra bundles.** In this section we define a group bundle G^{tw} and an associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ on an elliptic curve $X = X_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where τ belongs to the upper half plane $\mathfrak{H} := \{\mathrm{Im}\,\tau > 0\}$. We fix a global coordinate t on X which comes from that of $\mathbb{C}$.

Let G be the Lie group $SL_N(\mathbb{C})$ and $\mathfrak{g}$ be its Lie algebra,

$$sl_N(\mathbb{C}) = \{A \in M_N(\mathbb{C}) \mid \mathrm{tr}\,A = 0\}.$$

We fix an invariant inner product of $\mathfrak{g}$ by

$$(A|B) := \mathrm{tr}(AB) \quad \text{for } A, B \in \mathfrak{g}. \tag{1.1}$$

Define matrices α and β by

$$\alpha := \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} 1 & & & 0 \\ & \varepsilon & & \\ & & \ddots & \\ 0 & & & \varepsilon^{N-1} \end{pmatrix}, \tag{1.2}$$

where $\varepsilon = \exp(2\pi i/N)$. Then we have $\alpha^N = \beta^N = 1$ and $\alpha\beta = \varepsilon\beta\alpha$.

We define the group bundle G^{tw} and its associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ by

$$G^{\mathrm{tw}} :=(\mathbb{C} \times G)/\sim, \tag{1.3}$$

$$\mathfrak{g}^{\mathrm{tw}} :=(\mathbb{C} \times \mathfrak{g})/\approx, \tag{1.4}$$

where the equivalence relations $\sim$ and $\approx$ are defined by

$$(t, g) \sim (t + 1, \alpha g \alpha^{-1}) \sim (t + \tau, \beta g \beta^{-1}), \tag{1.5}$$

$$(t, A) \approx (t + 1, \alpha A \alpha^{-1}) \approx (t + \tau, \beta A \beta^{-1}). \tag{1.6}$$

(Because of $\alpha\beta = \varepsilon\beta\alpha$, the group bundle G^{tw} is *not* a principal bundle.) The fibers of G^{tw} are isomorphic to G and those of $\mathfrak{g}^{\mathrm{tw}}$ are isomorphic to $\mathfrak{g}$, but there are not canonical isomorphisms.

The twisted Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ has a natural connection, $\nabla_{d/dt} = d/dt$, and is decomposed into a direct sum of line bundles:

$$\mathfrak{g}^{\mathrm{tw}} \cong \bigoplus_{(a,b)\neq(0,0)} L_{a,b}, \tag{1.7}$$

where the indices (a, b) runs through $(\mathbb{Z}/N\mathbb{Z})^N \smallsetminus \{(0, 0)\}$. Here the line bundle $L_{a,b}$ on X is defined by

$$L_{a,b} := (\mathbb{C} \times \mathbb{C})/{\approx_{a,b}}, \tag{1.8}$$

where $\approx_{a,b}$ is an equivalence relation defined by

$$(t,x) \approx_{a,b} (t+1, \varepsilon^a x) \approx_{a,b} (t+\tau, \varepsilon^b x). \tag{1.9}$$

We regard $L_{a,b}$ as a line subbundle of $\mathfrak{g}^{\mathrm{tw}}$ through the injection given by

$$L_{a,b} \ni (t,x) \mapsto (t, xJ_{a,b}) \in \mathfrak{g}^{\mathrm{tw}}, \tag{1.10}$$

where $J_{a,b}$ is the element of $\mathfrak{g}$ defined by

$$J_{a,b} := \beta^a \alpha^{-b}. \tag{1.11}$$

We remark that $\{ J_{a,b} \mid (a,b) \in (\mathbb{Z}/N\mathbb{Z})^N \smallsetminus \{(0,0)\} \}$ is a basis of $\mathfrak{g} = sl_N(\mathbb{C})$.

The space of meromorphic sections of $L_{a,b}$ over X pulled back to $\mathbb{C}$ can be canonically identified with

$$K_{a,b} = \{ f \in \mathcal{M}(\mathbb{C}) \mid f(t+1) = \varepsilon^a f(t),\ f(t+\tau) = \varepsilon^b f(t) \}. \tag{1.12}$$

Here $\mathcal{M}(\mathbb{C})$ is the space of meromorphic functions on $\mathbb{C}$. ($K_{0,0}$ is the space of elliptic functions and corresponds to the trivial line bundle on X.) The mapping

$$f \mapsto (t, f(t)) \text{ modulo } \approx_{a,b} \tag{1.13}$$

gives a canonical isomorphism from $K_{a,b}$ onto $H^0(X, L_{a,b} \otimes \mathcal{K}_X)$.

The Liouville theorem implies that the only holomorphic function in $K_{a,b}$ is zero when $(a,b) \neq (0,0)$. This is equivalent to $H^0(X, L_{a,b}) = 0$. Since $L^*_{a,b} \cong L_{-a,-b}$ and the canonical line bundle of X is trivial, it follows from the Serre duality that $H^1(X, L_{a,b}) = 0$. Thus we obtain a simple vanishing result $H^p(X, L_{a,b}) = 0$ and therefore from the decomposition (1.7) we obtain the following result.

Lemma 1.1. $H^0(X, \mathfrak{g}^{\mathrm{tw}}) = H^1(X, \mathfrak{g}^{\mathrm{tw}}) = 0$.

Example 1.2. For $N = 2$, matrices α and β are nothing but the Pauli matrices σ^1 and σ^3. The Jacobian elliptic functions sn, cn, and dn are meromorphic functions in $K_{1,0}$, $K_{1,1}$, and $K_{0,1}$ and can be regarded as meromorphic sections of the line bundles $L_{1,0}$, $L_{1,1}$, and $L_{0,1}$ respectively.

Example 1.3. For general N and each $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0,0)\}$, we define the function $w_{a,b}$ by

$$w_{a,b}(t) = w_{a,b}(\tau; t) := \frac{\theta'_{[0,0]}}{\theta_{[a,b]}} \frac{\theta_{[a,b]}(t;\tau)}{\theta_{[0,0]}(t;\tau)}. \tag{1.14}$$

(See (A.8) in Appendix A for the notation.) The function $w_{a,b}(t)$ on $\mathbb{C}$ is uniquely characterized by the following properties:

1. The function $w_{a,b}(t)$ is a meromorphic functions in $K_{a,b}$ and hence can be regarded as a global meromorphic section of $L_{a,b}$;
2. The poles of $w_{a,b}(t)$ are all simple and contained in $\mathbb{Z} + \mathbb{Z}\tau$;
3. The residue of $w_{a,b}(t)$ at $t = 0$ is equal to 1.

Because of these properties, it will play an important role in concrete computations in later sections. For convenience of those computations, let us list several other properties of $w_{a,b}(t)$:

- The Laurent expansion at $t = 0$ is equal to

$$w_{a,b}(t) = \frac{1}{t} + \sum_{\nu=0}^{\infty} w_{a,b,\nu} t^{\nu} = \frac{1}{t} + w_{a,b,0} + w_{a,b,1} t + \cdots , \tag{1.15}$$

where the coefficients are written in the following forms:

$$w_{a,b,0} = \frac{\theta'_{[a,b]}}{\theta_{[a,b]}}, \quad w_{a,b,1} = \frac{\theta''_{[a,b]}}{2\theta_{[a,b]}} - \frac{\theta'''_{[0,0]}}{6\theta'_{[0,0]}}, \quad \dots \tag{1.16}$$

- Formulae (A.6) and (A.7) imply

$$w_{-a,-b}(t) = -w_{a,b}(-t), \quad w_{a,b}(t) = w_{a',b'}(t) \text{ if } a \equiv a' \text{ and } b \equiv b' \bmod N. \tag{1.17}$$

- Lemma A.1 will be used in the following form:

$$\sum_{(a,b)\neq(0,0)} w_{a,b,1} = 0, \tag{1.18}$$

where the summation is taken over all $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0, 0)\}$.

1.2. **Definition of the spaces of conformal coinvariants and conformal blocks.** In this section we define a conformal block associated to the twisted Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ defined in §1.1.

First let us introduce notation of sheaves. As usual, the structure sheaf on $X = X_\tau$ is denoted by $\mathcal{O}_X$ and the sheaf of meromorphic functions on X by $\mathcal{K}_X$. A stalk of a sheaf $\mathcal{F}$ on X at a point $P \in X$ is denoted by $\mathcal{F}_P$. When $\mathcal{F}$ is a $\mathcal{O}_X$-module, we denote its fiber $\mathcal{F}_P/\mathfrak{m}_P\mathcal{F}_P$ by $\mathcal{F}|_P$, where $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Denote by $\mathcal{F}_P^\wedge$ the $\mathfrak{m}_P$-adic completion of $\mathcal{F}_P$.

We shall use the same symbol for a vector bundle and for a locally free coherent $\mathcal{O}_X$-module consisting of its local holomorphic sections. For instance, the invertible sheaf associated to the line bundle $L_{a,b}$ is also denoted by the same symbol $L_{a,b}$. Denote by Ω_X^1 the sheaf of holomorphic 1-forms on X, which is isomorphic to $\mathcal{O}_X$ since X is an elliptic curve. The fiberwise Lie algebra structure of the bundle $\mathfrak{g}^{\mathrm{tw}}$ induces that of the associated sheaf $\mathfrak{g}^{\mathrm{tw}}$ over $\mathcal{O}_X$. Define the invariant $\mathcal{O}_X$-inner product on $\mathfrak{g}^{\mathrm{tw}}$ by

$$(A|B) := \frac{1}{2N} \operatorname{tr}_{\mathfrak{g}^{\mathrm{tw}}}(\operatorname{ad} A \operatorname{ad} B) \in \mathcal{O}_X \quad \text{for } A, B \in \mathfrak{g}^{\mathrm{tw}}, \tag{1.19}$$

where the symbol ad denotes the adjoint representation of the $\mathcal{O}_X$-Lie algebra $\mathfrak{g}^{\mathrm{tw}}$. Then the inner product on $\mathfrak{g}^{\mathrm{tw}}$ is invariant under the translations with respect to the connection $\nabla : \mathfrak{g}^{\mathrm{tw}} \to \mathfrak{g}^{\mathrm{tw}} \otimes_{\mathcal{O}_X} \Omega_X^1$:

$$d(A|B) = (\nabla A|B) + (A|\nabla B) \in \Omega_X^1 \quad \text{for } A, B \in \mathfrak{g}^{\mathrm{tw}}. \tag{1.20}$$

Under the trivialization of $\mathfrak{g}^{\mathrm{tw}}$ defined by the construction (1.4), the connection ∇ and the inner product $(\cdot|\cdot)$ on $\mathfrak{g}^{\mathrm{tw}}$ respectively coincide with the exterior derivative by t and the inner product defined by (1.1).

For any point P on X with $t(P) = z$, we put

$$\mathfrak{g}^P := (\mathfrak{g}^{\mathrm{tw}} \otimes_{\mathcal{O}_X} \mathcal{K}_X)_P^\wedge,$$

which is a topological Lie algebra non-canonically isomorphic to the loop algebra $\mathfrak{g}((t - z))$. Its subspace $\mathfrak{g}_+^P := (\mathfrak{g}^{\mathrm{tw}})_P^\wedge \cong \mathfrak{g}[[t - z]]$ is a maximal linearly compact subalgebra of $\mathfrak{g}^P$ under the $(t - z)$-adic linear topology.

Let us fix mutually distinct points $Q_1, \dots, Q_L$ on X whose coordinates are $t = z_1, \dots, z_L$ and put $D := \{Q_1, \dots, Q_L\}$. We shall also regard D as a divisor on X (i.e., $D = Q_1 + \cdots + Q_L$). Denote $X \smallsetminus D$ by $\dot{X}$. The Lie algebra $\mathfrak{g}^D := \bigoplus_{i=1}^L \mathfrak{g}^{Q_i}$ has the natural 2-cocycle defined by

$$\mathrm{c_a}(A, B) := \sum_{i=1}^L \operatorname*{Res}_{t=z_i} (\nabla A_i | B_i), \tag{1.21}$$

where $A = (A_i)_{i=1}^L, B = (B_i)_{i=1}^L \in \mathfrak{g}^D$ and $\mathrm{Res}_{t=z}$ is the residue at $t = z$. (The symbol "$\mathrm{c_a}$" stands for "Cocycle defining the Affine Lie algebra".) We denote the central extension of $\mathfrak{g}^D$ with respect to this cocycle by $\hat{\mathfrak{g}}^D$:

$$\hat{\mathfrak{g}}^D := \mathfrak{g}^D \oplus \mathbb{C}\hat{k},$$

where $\hat{k}$ is a central element. Explicitly the bracket of $\hat{\mathfrak{g}}^D$ is represented as

$$[A, B] = ([A_i, B_i]^0)_{i=1}^L \oplus \mathrm{c_a}(A, B)\hat{k} \quad \text{for } A, B \in \mathfrak{g}^D, \tag{1.22}$$

where $[A_i, B_i]^0$ are the natural bracket in $\mathfrak{g}^{Q_i}$. The Lie algebra $\hat{\mathfrak{g}}^P$ for a point P is non-canonically isomorphic to the affine Lie algebra $\hat{\mathfrak{g}}$ of type $A_{N-1}^{(1)}$ (a central extension of the loop algebra $\mathfrak{g}((t - z)) = sl_N\bigl(\mathbb{C}((t - z))\bigr)$). If $P = Q_i$ for $i = 1, \dots, L$, then $\hat{\mathfrak{g}}^P = \hat{\mathfrak{g}}^{Q_i}$ can be regarded as a subalgebra of $\hat{\mathfrak{g}}^D$. Put $\mathfrak{g}_+^P := (\mathfrak{g}^{\mathrm{tw}})_P^\wedge$ as above. Then $\mathfrak{g}_+^{Q_i}$ can be also regarded as a subalgebra of $\hat{\mathfrak{g}}^{Q_i}$ and $\hat{\mathfrak{g}}^D$.

Let $\mathfrak{g}_{\dot{X}}^D$ be the space of global meromorphic sections of $\mathfrak{g}^{\mathrm{tw}}$ which are holomorphic on $\dot{X}$:

$$\mathfrak{g}_{\dot{X}}^D := \Gamma(X, \mathfrak{g}^{\mathrm{tw}}(*D)).$$

There is a natural linear map from $\mathfrak{g}_{\dot{X}}^D$ into $\mathfrak{g}^D$ which maps a meromorphic section of $\mathfrak{g}^{\mathrm{tw}}$ to its germ at Q_i's. As in the non-twisted case (e.g., §2.2 of [TUY]), the residue theorem implies that this linear map is extended to a Lie algebra injection from $\mathfrak{g}_{\dot{X}}^D$ into $\hat{\mathfrak{g}}^D$, which allows us to regard $\mathfrak{g}_{\dot{X}}^D$ as a subalgebra of $\hat{\mathfrak{g}}^D$.

Definition 1.4. The space of *conformal coinvariants* $\mathrm{CC}(M)$ and that of *conformal blocks* $\mathrm{CB}(M)$ associated to $\hat{\mathfrak{g}}^{Q_i}$-modules M_i with the same level $\hat{k} = k$ are defined to be the space of coinvariants of $M := \bigotimes_{i=1}^L M_i$ with respect to $\mathfrak{g}_{\dot{X}}^D$ and its dual:

$$\mathrm{CC}(M) := M/\mathfrak{g}_{\dot{X}}^D M, \quad \mathrm{CB}(M) := (M/\mathfrak{g}_{\dot{X}}^D M)^*. \tag{1.23}$$

(In [TUY], $\mathrm{CC}(M)$ and $\mathrm{CB}(M)$ are called the space of *covacua* and that of *vacua* respectively.) In other words, the space of conformal coinvariants $\mathrm{CC}(M)$ is generated by M with relations

$$A_{\dot{X}} v \equiv 0 \tag{1.24}$$

for all $A_{\dot{X}} \in \mathfrak{g}_{\dot{X}}^D$ and $v \in M$, and a linear functional Φ on M belongs to the space of conformal blocks $\mathrm{CB}(M)$ if and only if it satisfies that

$$\Phi(A_{\dot{X}} v) = 0 \quad \text{for all } A_{\dot{X}} \in \mathfrak{g}_{\dot{X}}^D \text{ and } v \in M. \tag{1.25}$$

These equations (1.24) and (1.25) are called the *Ward identities.*

The most important conformal blocks for our purpose are constructed from Weyl modules (or generalized Verma modules) which are determined from the following data:

- A parameter $k \in \mathbb{C}$ which is called the level of the model.
- Finite-dimensional irreducible representations V_i of the fiber Lie algebra $\mathfrak{g}^{\mathrm{tw}}|_{Q_i}$ isomorphic to $\mathfrak{g}$.

Put $\mathfrak{g}_+^{Q_i} := (\mathfrak{g}^{\mathrm{tw}})^\wedge_{Q_i}$ and $\hat{\mathfrak{g}}_+^{Q_i} := \mathfrak{g}_+^{Q_i} \oplus \mathbb{C}\hat{k}$, which are subalgebras of $\hat{\mathfrak{g}}^{Q_i}$. The subalgebra $\hat{\mathfrak{g}}_+^{Q_i} := \mathfrak{g}_+^{Q_i} \oplus \mathbb{C}\hat{k}$ of $\hat{\mathfrak{g}}^{Q_i}$ acts on V_i through the linear map $\hat{k} \mapsto k\,\mathrm{id}_{V_i}$ and the natural projection $\mathfrak{g}_+^{Q_i} \to \mathfrak{g}^{\mathrm{tw}}|_{Q_i}$, $A \mapsto A(Q_i)$. The $\hat{\mathfrak{g}}^{Q_i}$-module induced from V_i is called a *Weyl module* or a *generalized Verma module*:

$$M_k(V_i) := \mathrm{Ind}_{\hat{\mathfrak{g}}_+^{Q_i}}^{\hat{\mathfrak{g}}^{Q_i}} V_i = U(\hat{\mathfrak{g}}^{Q_i}) \otimes_{U(\hat{\mathfrak{g}}_+^{Q_i})} V_i \tag{1.26}$$

See [KL] for properties of Weyl modules.

The space of conformal coinvariants and that of conformal blocks associated to the data $(Q,V) = (\{Q_i\},\{V_i\})$ are defined to be the space of conformal coinvariants and that of conformal blocks associated to the $\hat{\mathfrak{g}}^D$-module

$$M_k(V) := \bigotimes_{i=1}^{L} M_k(V_i),$$

on which the center $\hat{k}$ acts as multiplication by k. Namely we define them as follows:

$$\mathrm{CC}_k(Q,V) = \mathrm{CC}_k(\{Q_i\},\{V_i\}) := \mathrm{CC}(M_k(V)) = M_k(V)/\mathfrak{g}_X^D M_k(V), \tag{1.27}$$

$$\mathrm{CB}_k(Q,V) = \mathrm{CB}_k(\{Q_i\},\{V_i\}) := \mathrm{CB}(M_k(V)) = (M_k(V)/\mathfrak{g}_X^D M_k(V))^*. \tag{1.28}$$

Hereafter we use the word “conformal coinvariants” and “conformal block” for this kind of conformal coinvariants and conformal blocks, namely those associated to Weyl modules, unless otherwise stated.

It is easy to see that the spaces of conformal coinvariants and conformal blocks are determined by the finite-dimensional part $V = \bigotimes_{i=1}^L V_i$ of $M_k(V)$ as is the case with the space of conformal coinvariants and conformal blocks on $\mathbb{P}^1(\mathbb{C})$ (e.g., Lemma 1 of [FFR]), because of the cohomology vanishing. In fact, Lemma 1.1 implies a decomposition,

$$\hat{\mathfrak{g}}^D = \mathfrak{g}_X^D \oplus \hat{\mathfrak{g}}_+^D, \tag{1.29}$$

where $\hat{\mathfrak{g}}_+^D = \bigoplus_{i=1}^L \mathfrak{g}_+^{Q_i} \oplus \mathbb{C}\hat{k}$. Hence we have, as left $\mathfrak{g}_X^D$-modules,

$$M_k(V) = \mathrm{Ind}_{\hat{\mathfrak{g}}_+^D}^{\hat{\mathfrak{g}}^D} V = U(\hat{\mathfrak{g}}^D) \otimes_{U(\hat{\mathfrak{g}}_+^D)} V = U(\mathfrak{g}_X^D) \otimes_{\mathbb{C}} V, \tag{1.30}$$

where $V := \bigotimes_{i=1}^L V_i$ and the action of $\hat{\mathfrak{g}}_+^D$ is defined by the mapping $\hat{k} \mapsto k \cdot \mathrm{id}$ and the natural projection $\mathfrak{g}_+^D \to \prod_{i=1}^L (\mathfrak{g}^{\mathrm{tw}}|_{Q_i})$. Therefore, due to the Ward identity (1.25) and the definition of the Weyl module (1.26), the space of conformal coinvariants is canonically isomorphic to the tensor product of $\mathfrak{g}^{\mathrm{tw}}|_{Q_i}$-modules by the natural inclusion map $V = \bigotimes_{i=1}^L V_i \hookrightarrow M_k(V)$, $\bigotimes_{i=1}^L v_i \mapsto \bigotimes_{i=1}^L (1 \otimes v_i)$. (In the following we shall identify $v_i \in V_i$ with $1 \otimes v_i \in M_k(V_i)$.)

Proposition 1.5. *The inclusion map $V \hookrightarrow M_k(V)$ and the induced restriction map $M_k(V)^* \twoheadrightarrow V^*$ induce the following isomorphisms respectively:*

$$\mathrm{CC}_k(Q,V) \xleftarrow{\sim} V = \bigotimes_{i=1}^L V_i \quad \textit{and} \quad \mathrm{CB}_k(Q,V) \xrightarrow{\sim} V^* = \bigotimes_{i=1}^L V_i^*.$$

For any point $P \in X = X_\tau$, Let us denote the 1-dimensional trivial representation of $\mathfrak{g}^{\mathrm{tw}}|_P$ by $\mathbb{C}_P = \mathbb{C}u_P$. Then the proposition above readily leads to the following corollary.

Corollary 1.6. *Let P be a point of X distinct from Q_i's. Then the canonical inclusion $M_k(V) \hookrightarrow M_k(\mathbb{C}_P) \otimes M_k(V)$, $v \mapsto u_P \otimes v$, induces an isomorphism*

$$\mathrm{CB}_k(\{P, Q_i\}, \{\mathbb{C}_P, V_i\}) \xrightarrow{\sim} \mathrm{CB}_k(\{Q_i\}, \{V_i\}). \tag{1.31}$$

The property above is called *propagation of vacua* in [TUY]. In our case the proof is far simpler due to Proposition 1.5, as in the case of $\mathbb{P}^1$. (cf. §3 of [FFR].)

1.3. **Correlation functions.** The current and the energy-momentum correlation functions are defined as in §2 of [TUY], but we must take twisting into account and use the decomposition (1.7).

First we consider the current correlation functions. Let Φ be a conformal block in $\mathrm{CB}_k(Q, V)$ and v a vector in $M_k(V)$. There exists a unique $\omega_i \in (L^*_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X \otimes_{\mathcal{O}_X} \Omega^1_X)^\wedge_{Q_i}$ with the property that

$$\mathrm{Res}_{t=z_i}\langle f_i, \omega_i\rangle := \Phi(\rho_i(f_i J_{a,b})v) \quad \text{for all } f_i \in (\mathcal{K}_X)^\wedge_{Q_i}, \tag{1.32}$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing of $(L_{a,b})^\wedge_P$ and $(L^*_{a,b})^\wedge_P$, $f_i J_{a,b}$ can be regarded as an element of $\mathfrak{g}^{Q_i}$ by means of (1.10) and its action on the i-th component of v (namely, the $M_k(V_i)$-component) is denoted by $\rho_i(f_i J_{a,b})$. Thus we obtain a linear functional

$$\sum_{i=1}^{L} \mathrm{Res}_{Q_i}\langle\cdot, \omega_i\rangle : \bigoplus_{i=1}^{L} (L_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X)^\wedge_{Q_i} \to \mathbb{C}$$

which maps $(f_i)_{i=1}^L \in (L_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X)^\wedge_{Q_i}$ to $\sum_{i=1}^L \mathrm{Res}_{Q_i}\langle f_i, \omega_i\rangle \in \mathbb{C}$.

The Ward identity (1.25) implies that

$$\sum_{i=1}^{L} \mathrm{Res}_{Q_i}\langle f_{Q_i}, \omega_i\rangle = \sum_{i=1}^{L} \Phi(\rho_i(f_{Q_i} J_{a,b})v) = 0$$

for any meromorphic section $f \in H^0(X, L_{a,b}(*D))$, where f_{Q_i} is the germ of f at Q_i. Since $H^0(X, L_{a,b}(*D))$ and $H^0(X, L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D))$ are orthogonal complements to each other under the residue pairing of $\bigoplus_{i=1}^L (L_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X)^\wedge_{Q_i}$ and $\bigoplus_{i=1}^L (L^*_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X \otimes \Omega^1_X)^\wedge_{Q_i}$ (cf. [Tat] or Theorem 2.20 of [I]), we have a meromorphic 1-form ω with values in $L^*_{a,b}$ such that the germ of ω at Q_i gives ω_i and is holomorphic outside of $\{Q_1, \dots, Q_L\}$:

$$\omega \in H^0(X, L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D)), \quad (\omega)_{Q_i} = \omega_i. \tag{1.33}$$

In order to define the correlation functions, we need explicit expression of the action of $(\mathfrak{g}^{\mathrm{tw}})^\wedge_P$ which we can identify with the affine Lie algebra by fixing a trivialization of $\mathfrak{g}^{\mathrm{tw}}$ around P.

Let P be any point of X and $z(P)$ a point of $\mathbb{C}$ whose image in $X = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is equal to P. The description (1.4) of $\mathfrak{g}^{\mathrm{tw}}$ naturally determines a local trivialization of $\mathfrak{g}^{\mathrm{tw}}$ at P, once we fix the coordinate $t = z(P)$ of P. By means of this trivialization, we fix isomorphisms $\mathfrak{g}^P \cong \hat{\mathfrak{g}}$, $\mathfrak{g}^{\mathrm{tw}}|_P \cong \mathfrak{g}$, and so on. The induced trivialization of $L_{a,b}$ at P is the same as the trivialization defined by the isomorphism (1.13):

$$(L_{a,b})_P \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}, z(P)} \xleftarrow{\sim} \mathcal{O}_{X,P} \quad \text{for } (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2, \tag{1.34}$$

which corresponds ∇-flat sections of $L_{a,b}$ to constant functions on X. The decomposition (1.7) of $\mathfrak{g}^{\mathrm{tw}}$ induces a decomposition of its stalk at P and is consistent with the trivializations above:

$$\mathfrak{g}^{\mathrm{tw}}_P = \bigoplus J_{a,b}(L_{a,b})_P \cong \bigoplus J_{a,b}\mathcal{O}_{X,P} = \mathfrak{g} \otimes \mathcal{O}_{X,P}, \tag{1.35}$$

where the indices (a,b) run through $(\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0,0)\}$.

Let ξ be a local coordinate at P. For $A \in \mathfrak{g}$ and $m \in \mathbb{Z}$, we denote by $A[m]$ the element of $(\mathfrak{g}^{\mathrm{tw}})^{\wedge}_P \subset \hat{\mathfrak{g}}^P$ which is represented by $A\xi^m$ under the trivialization (1.35). Since $A[0]$ is ∇-flat for $A \in \mathfrak{g}$ (i.e., $\nabla A[0] = 0$), the bracket of $\hat{\mathfrak{g}}^P$ is represented as:

$$[A[m], B[n]] = [A,B][m+n] + (A|B) m \delta_{m+n,0} \hat{k} \quad \text{for } A, B \in \mathfrak{g} \text{ and } m, n \in \mathbb{Z},$$

which coincides with the usual commutation relation of the affine Lie algebra.

Lemma 1.7. *Under the situation above, let P be in $\dot{X}$ (i.e., distinct from Q_i's) and $\tilde{\Phi}$ the conformal block in* $\mathrm{CB}_k(\{P, Q_i\}, \{\mathbb{C}_P, V_i\})$ *corresponding to Φ by the isomorphism* (1.31). *Then we have the following:*
(i) *Take $x \in L_{a,b}|_P$ and let f_x be an element of $(L_{a,b})^{\wedge}_P$ with a principal part x/ξ (i.e., $f_x = (x/\xi + \mathrm{regular})$). Then $\tilde{\Phi}(f_x J_{a,b} u_P \otimes v)\, d\xi$ does not depend on the choice of ξ and f_x. Thus we can define $\tilde{\Phi}(J_{a,b}[-1] u_P \otimes v)\, d\xi \in (L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X)|_P$ by*

$$\langle x, \tilde{\Phi}(J_{a,b}[-1] u_P \otimes v)\rangle\, d\xi := \tilde{\Phi}(f_x J_{a,b} u_P \otimes v)\, d\xi.$$

(ii) *The following equation holds at any point P in $\dot{X} = X \smallsetminus \{Q_1, \dots, Q_L\}$:*

$$\omega(P) = \tilde{\Phi}(J_{a,b}[-1] u_P \otimes v)\, d\xi \in (L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X)|_P. \tag{1.36}$$

Proof. The statement (i) can be shown by the same argument as the proof of Claim 1 of Theorem 2.4.1 of [TUY]. Using the Riemann-Roch theorem (or the function $w_{a,b}(t)$ defined by (1.14)), we can choose f_x in (i) from $H^0(X, L_{a,b}(*D + P))$. Then we have

$$\begin{aligned}
&\langle x, \tilde{\Phi}(J_{a,b}[-1] u_P \otimes v)\rangle\, d\xi = \tilde{\Phi}(f_x J_{a,b} u_P \otimes v)\, d\xi \\
&= -\sum_{i=1}^{L} \tilde{\Phi}(u_P \otimes \rho_i(f_x J_{a,b}) v)\, d\xi \\
&= -\sum_{i=1}^{L} \Phi(\rho_i(f_x J_{a,b}) v)\, d\xi = -\sum_{i=1}^{L} \mathrm{Res}_{Q_i} \langle f_x, \omega\rangle\, d\xi \\
&= \mathrm{Res}_P \langle f_x, \omega\rangle\, d\xi = \mathrm{Res}_P \frac{\langle x, \omega\rangle}{\xi}\, d\xi = \langle x, \omega|_P\rangle.
\end{aligned}$$

Here we have used the Ward identity (1.25) and the residue theorem. Thus we have proved the equation (1.36). □

Definition 1.8. We call this 1-form $\omega = \tilde{\Phi}(J_{a,b}[-1] u_P \otimes v)\, d\xi$ a *correlation function of the current $J_{a,b}(\xi)$* and v under Φ, or a *current correlation function* for short, and denote it by $\Phi(J_{a,b}(P) v)\, dP$ or $\Phi(J_{a,b}(\xi) v)\, d\xi$ when we fix a local coordinate ξ.

We now proceed to the definition of the energy-momentum correlation functions.

Lemma 1.9. *Let P be in $\dot{X}$ and ξ a local coordinate defined on an open neighborhood U of P. Then the following expression gives a holomorphic section of $\Omega^2_X(*D) = (\Omega^1_X)^{\otimes 2}(*D)$ on sufficiently small U:*

$$
\begin{aligned}
&\Phi(S(P)v)\,(d\xi(P))^2 \\
&\quad := \frac{1}{2}\lim_{P'\to P}\left(\sum_{(a,b)} \Phi(J_{a,b}(\xi(P))\, J^{a,b}(\xi(P'))v) \right.\\
&\qquad\qquad \left. - \frac{k \dim \mathfrak{g}}{(\xi(P)-\xi(P'))^2}\right) d\xi(P)\, d\xi(P'). \qquad (1.37)
\end{aligned}
$$

Here the indices (a,b) run through $(\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0,0)\}$, and $J^{a,b}$ is the dual basis of $J_{a,b}$ with respect to $(\cdot|\cdot)$, namely, $J^{a,b} = J_{-a,-b}/N$.

Proof. The argument in the proof of Lemma 1.7 and the Hartogs theorem of holomorphy show that the current correlation function $\Phi(J_{a,b}(\xi)J^{a,b}(\zeta)v)\, d\xi\, d\zeta$ defines a global section on $X \times X$ of the sheaf $\mathcal{F} := (L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D)) \boxtimes (L^*_{-a,-b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D))(*\Delta)$, where Δ is the diagonal divisor of $X \times X$. We define

$$
\Phi(S(P,P')v)\, d\xi(P)\, d\xi(P') \in H^0(U \times U, \mathcal{F})
$$

by

$$
\begin{aligned}
&\Phi(S(P,P')v)\, d\xi(P)\, d\xi(P') \\
&\quad := \left(\sum_{a,b} \Phi(J_{a,b}(\xi(P))\, J^{a,b}(\xi(P'))v) - \frac{k \dim \mathfrak{g}}{(\xi(P)-\xi(P'))^2}\right) d\xi(P)\, d\xi(P').
\end{aligned}
$$

In order to show Lemma 1.9, first take a local coordinate ξ on a sufficiently small neighborhood U of P with $\xi(P)=0$ and a local trivialization of $L_{a,b}$ on U and choose a meromorphic section $f \in H^0(X, L_{a,b} \otimes_{\mathcal{O}_X} \mathcal{K}_X)$ whose Laurent expansion has the form

$$
f(\xi) = \xi^{-1} + (\text{regular at } \xi = 0). \qquad (1.38)
$$

The inclusion (1.10) and the Ward identity (1.25) imply that $\Phi(J_{a,b}(\xi(P))J^{a,b}(\xi(P'))v)$ is equal to

$$
\begin{aligned}
&\tilde{\tilde{\Phi}}(J_{a,b}[-1]u_P \otimes J^{a,b}[-1]u_{P'} \otimes v) \\
&\quad = -\tilde{\Phi}((fJ_{a,b})_{P'} J^{a,b}[-1]u_{P'} \otimes v) - \sum_{i=1}^{L} \tilde{\Phi}(J^{a,b}[-1]u_{P'} \otimes \rho_i((fJ_{a,b})_{Q_i})v), \qquad (1.39)
\end{aligned}
$$

where $\tilde{\tilde{\Phi}} \in \mathrm{CB}_k(\{P,P',Q_i\},\{\mathbb{C}_P,\mathbb{C}_{P'},V_i\})$, and $\tilde{\Phi} \in \mathrm{CB}_k(\{P',Q_i\},\{\mathbb{C}_{P'},V_i\})$ correspond to Φ through the isomorphism (1.31). The second term of (1.39) is regular as a function of P at P' as shown in Lemma 1.7 and the first term is rewritten as

$$
-\tilde{\Phi}((fJ_{a,b})_{P'} J^{a,b}[-1]u_{P'} \otimes v) = \frac{k}{(\xi(P)-\xi(P'))^2}\Phi(v) + (\text{regular at } P' = P). \qquad (1.40)
$$

(Details of the computation is the same as that of the proof of the assertion (4) of Theorem 2.4.1 of [TUY]. Note that $[J_{a,b}, J^{a,b}] = 0$.) Equations (1.39) and (1.40) mean that $\Phi(S(P,P')v)\, d\xi(P)\, d\xi(P')$ is a holomorphic section of $(L^*_{a,b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D)) \boxtimes (L^*_{-a,-b} \otimes_{\mathcal{O}_X} \Omega^1_X(*D))$ on $U \times U$ for the coordinate ξ that satisfies (1.38).

Restricting it to the diagonal of $U \times U$, we obtain a local holomorphic section

$$\Phi(S(P)v)\,(d\xi(P))^2 \in H^0(U, \Omega_X^2(*D)).$$

Note that since $L^*_{a,b} = L_{-a,-b}$, the factors $L^*_{a,b}$ and $L^*_{-a,-b}$ cancel out on the diagonal. Thanks to this fact, the trivialization of $L_{a,b}$'s which we implicitly fixed in the argument above does not affect the result. □

Definition 1.10. Put $\kappa := k + h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (i.e., $h^\vee = N$ because $\mathfrak{g} = sl_N$). We call $\Phi(S(\xi)v)\,(d\xi)^2$ in (1.37) a *correlation function of the Sugawara tensor* $S(\xi)$ and v under Φ, or a *Sugawara correlation function* for short, and

$$\Phi(T(\xi)v)\,(d\xi)^2 := \kappa^{-1}\Phi(S(\xi)v)\,(d\xi)^2 \tag{1.41}$$

a *correlation function of the energy-momentum tensor* $T(\xi)$ and v under Φ, or a *energy-momentum correlation function* for short when $\kappa \neq 0$.

Let us calculate the coordinate transformation law of the correlation function of the Sugawara tensor. Let ζ be another coordinate on U. The differential $d\xi(P)d\xi(P')/(\xi(P)-\xi(P'))^2$ transforms under the coordinate change $\xi \mapsto \zeta = \zeta(\xi)$ as

$$\frac{d\zeta(P)\,d\zeta(P')}{(\zeta(P)-\zeta(P'))^2} = \frac{d\xi(P)\,d\xi(P')}{(\xi(P)-\xi(P'))^2} + \frac{\{\zeta,\xi\}(P')}{6}\,d\xi(P)\,d\xi(P') + O(\xi(P)-\xi(P')), \tag{1.42}$$

where $\{\zeta,\xi\} = \zeta'''/\zeta' - 3/2(\zeta''/\zeta')^2$ ($\zeta' = d\zeta/d\xi$) is the Schwarzian derivative of $\zeta = \zeta(P)$ with respect to $\xi = \xi(P)$. Hence the correlation function of the Sugawara tensor transforms with respect to a coordinate change $\xi \mapsto \zeta = \zeta(\xi)$ not as 2-differentials but as

$$\Phi(S(\zeta)v)d\zeta^2 = \Phi(S(\xi)v)d\xi^2 + \frac{k \dim \mathfrak{g}}{12}\{\zeta,\xi\}\Phi(v). \tag{1.43}$$

This means that the family $\{\Phi(S(\xi)v)d\xi^2\}$ defines a meromorphic projective connection on X. (For the notion of projective connections on Riemann surfaces, see [Gu].) The Schwarzian derivative $\{\zeta,\xi\}$ vanishes identically if and only if ζ is a fractional linear transformation of ξ (i.e., $\zeta = (a\xi+b)/(c\xi+d)$). From this fact it follows that $\{\Phi(S(\xi)v)\,d\xi^2\}$ behaves like a 2-differential under fractional linear coordinate changes.

For later use, we compute the local expression of the energy-momentum correlation function around Q_i. We take a holomorphic local chart (U,ζ) with $Q_i \in U$ and $\zeta(Q_i) = 0$, a local trivialization of $L_{a,b}$ by (1.34), and a trivialization of $\mathfrak{g}^{\mathrm{tw}}_{Q_i}$ by (1.35). Under these trivializations, we have, due to (1.33),

$$\begin{aligned}\Phi(J_{a,b}(P_1)J^{a,b}(P_2)v)\,dP_1\,dP_2 \\ = \Biggl(\sum_{i=1}^{L}\sum_{n,m\in\mathbb{Z}} \zeta_1^{-m-1}\zeta_2^{-n+m-1}\Phi(\rho_i({}^\circ_\circ J^{a,b}[n-m]J_{a,b}[m]{}^\circ_\circ)v) \\ + \frac{k\Phi(v)}{(\zeta_2-\zeta_1)^2}\Biggr)\,d\zeta_1\,d\zeta_2\end{aligned} \tag{1.44}$$

if $|\zeta_1| > |\zeta_2|$. Here $P_1, P_2 \in U$, $\zeta_1 = \zeta(P_1)$, $\zeta_2 = \zeta(P_2)$, and $\substack{\circ \\ \circ} \quad \substack{\circ \\ \circ}$ denotes the normal ordered product defined by

$$\substack{\circ \\ \circ} A[m]B[n] \substack{\circ \\ \circ} = \begin{cases} A[m]B[n], & \text{if } m < n, \\ \frac{1}{2}(A[m]B[n] + B[n]A[m]), & \text{if } m = n, \\ B[n]A[m], & \text{if } m > n. \end{cases} \tag{1.45}$$

Using (1.44), we obtain an expression of the correlation function (1.37) around the point Q_i:

$$\Phi(S(\zeta)v)(d\xi)^2 = \sum_{m\in\mathbb{Z}} \zeta^{-m-2} \Phi(\rho_i(S[m])v)\,(d\zeta)^2, \tag{1.46}$$

where $S[m]$ are the *Sugawara operators* defined by:

$$S[m] = \frac{1}{2} \sum_{a,b} \sum_{n\in\mathbb{Z}} \substack{\circ \\ \circ} J_{a,b}[m-n]\, J^{a,b}[n] \substack{\circ \\ \circ}, \tag{1.47}$$

which satisfy the following commutation relations:

$$[S[m], A[n]] = -\kappa n A[m+n] \quad \text{for } A \in \mathfrak{g}, \tag{1.48}$$

$$[S[m], S[n]] = \kappa \left((m-n)S[m+n] + \frac{k \dim \mathfrak{g}}{12}(m^3 - m)\delta_{m+n,0}\,\mathrm{id} \right) \tag{1.49}$$

In particular the Sugawara operators $S[m]$ commute with $\mathfrak{g}^P$ if $\kappa = 0$ (i.e., the level k is critical).

When $\kappa = k + h^\vee \neq 0$, the usual Virasoro operators are defined by normalizing $S[m]$:

$$T[m] := \kappa^{-1} S[m], \tag{1.50}$$

which satisfy the well-known commutation relations:

$$[T[m], A[n]] = -nA[m+n] \quad \text{for } A \in \mathfrak{g}, \tag{1.51}$$

$$[T[m], T[n]] = (m-n)T[m+n] + \frac{c_k}{12}(m^3 - m)\delta_{m+n,0}\,\mathrm{id}, \tag{1.52}$$

where $c_k = k \dim \mathfrak{g}/\kappa$.

Later we fix the local coordinate at Q_i to $\xi_i = t - z_i$ and the one at $P \in X$ to $\xi = t - z$ with $t(P) = z$, where t is the global coordinate of $\mathbb{C}$ (cf. (1.4)) and $z_i = t(Q_i)$.

Lemma 1.11. *In this coordinate $\Phi(S(\xi)v)\,(d\xi)^2$ and $\Phi(T(\xi)v)(d\xi)^2$ can be extended to a global 2-differentials $\Phi(S(t)v)(dt)^2$ and $\Phi(T(t)v)\,(dt)^2$.*

Proof. Under fractional linear coordinate changes, $\{\Phi(S(\xi)v)\,(d\xi)^2\}$ behaves like a 2-differential due to (1.43). Since the coordinate changes between two of ξ's are merely translations, if X is covered by these coordinates, then $\{\Phi(S(\xi)v)\,(d\xi)^2\}$ gives a meromorphic 2-differentials on X. □

Remark 1.12. Using Weierstraß' $\wp$-function, we can prove the lemma above in a more explicit manner. In fact, since

$$\wp(t_1 - t_2)\,dt_1 dt_2 = \left(\frac{1}{(t_1 - t_2)^2} + O((t_1 - t_2)^2) \right) dt_1\, dt_2$$

is a global meromorphic 2-form on $X \times X$ with a pole along the diagonal, we can equivalently replace the definition (1.37) by

$$\begin{aligned}
&\Phi(S(P)v)\,(dt(P))^2 \\
&\qquad := \frac{1}{2}\lim_{P'\to P}\Biggl(\sum_{a,b}\Phi(J_{a,b}(t(P))J^{a,b}(t(P'))v) \\
&\qquad\qquad - k\dim\mathfrak{g}\cdot\wp(t(P)-t(P'))\Biggr)\,dt(P)\,dt(P'). \qquad (1.53)
\end{aligned}$$

This definition is meaningful globally on X and coincides with that of the proof above.

1.4. **Action of the Virasoro algebra.** When the level k is not $-h^\vee$, the Lie algebras of formal meromorphic vector fields at Q_i are projectively represented on $M_k(V)$ through the energy-momentum tensor.

We denote by $\mathcal{T}_X$ the tangent sheaf of $X = X_\tau$ (i.e., the sheaf of vector fields on X). Let us fix a local coordinate at Q_i to $\xi_i = t - z_i$, and denote by $\mathcal{T}^D$ the direct sum of Lie algebra of formal meromorphic vector fields at Q_i for $i = 1, \dots, L$:

$$\mathcal{T}^D := \bigoplus_{i=1}^L \mathcal{T}^{Q_i}, \quad \mathcal{T}^{Q_i} := (\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{K}_X)^\wedge_{Q_i} = \bigoplus_{i=1}^L \mathbb{C}((\xi_i))\frac{\partial}{\partial\xi_i}. \qquad (1.54)$$

The *Virasoro algebra* $\mathcal{V}ir^D$ at D is defined to be the central extension of $\mathcal{T}^D$ by $\mathbb{C}\hat{c}$:

$$\mathcal{V}ir^D := \mathcal{T}^D \oplus \mathbb{C}\hat{c}, \qquad (1.55)$$

whose Lie algebra structure is defined by

$$\begin{aligned}
&\Bigl[\bigl(\theta_i(\xi_i)\partial_{\xi_i}\bigr)_{i=1}^L, \bigl(\eta_i(\xi_i)\partial_{\xi_i}\bigr)_{i=1}^L\Bigr] \\
&\quad = \Bigl((\theta_i(\xi_i)\eta_i'(\xi_i) - \eta_i(\xi_i)\theta_i'(\xi_i))\partial_{\xi_i}\Bigr)_{i=1}^L \oplus \frac{\hat{c}}{12}\sum_{i=1}^L \operatorname*{Res}_{\xi_i=0}(\theta_i'''(\xi_i)\eta_i(\xi_i)\,d\xi_i), \qquad (1.56)
\end{aligned}$$

where $\theta_i(\xi_i), \eta_i(\xi_i) \in \mathbb{C}((\xi_i))$ and $\partial_{\xi_i} := \partial/\partial\xi_i$. When $L = 1$, this Virasoro algebra $\mathcal{V}ir^D$ is the usual one defined as a central extension of the Lie algebra of vector fields on a circle.

The action of $\theta_i = \theta_i(\xi_i)\partial_{\xi_i} \in \mathcal{T}^{Q_i}$ and that of $\theta = (\theta_i(\xi_i)\partial_{\xi_i})_{i=1}^L \in \mathcal{T}^D$ on $v \in M_k(V)$ and $\Phi \in (M_k(V))^*$ are given by

$$T_i\{\theta_i\}v := -\sum_{m\in\mathbb{Z}}\rho_i(\theta_{i,m}T[m])v, \quad T\{\theta\}(v) := \sum_{i=1}^L T_i\{\theta_i\}v, \qquad (1.57)$$

$$(T_i^*\{\theta_i\}\Phi)(v) := -\Phi(T_i\{\theta_i\}v), \qquad T^*\{\theta\}\Phi := \sum_{i=1}^L T_i^*\{\theta_i\}\Phi, \qquad (1.58)$$

where $\theta_i(\xi_i)\partial_{\xi_i} = \sum_{m\in\mathbb{Z}}\theta_{i,m}\xi_i^{m+1}\partial_{\xi_i}$ is the Laurent expansion (cf. (1.47) and (1.50)). Note that we have

$$(T_i^*\{\theta_i\}\Phi)(v) = \operatorname*{Res}_{\xi_i=0}\langle\Phi(T(\xi_i)v)\,(d\xi_i)^2, \theta_i(\xi_i)\partial_{\xi_i}\rangle, \qquad (1.59)$$

where $\langle\cdot,\cdot\rangle$ is the contraction of a 2-differential and a tangent vector.

Proposition 1.13. *Let $\theta = (\theta_i \partial_{\xi_i})_{i=1}^L$ and $\eta = (\eta_i \partial_{\xi_i})_{i=1}^L$ be elements of $\mathcal{T}^D$. Then the operators $T\{\theta\}$, $T\{\eta\}$ acting on $M_k(V)$ and the operators $T^*\{\theta\}$, $T^*\{\eta\}$ acting on $(M_k(V))^*$ satisfy*

$$[T\{\theta\}, T\{\eta\}] = T\{[\theta, \eta]\} + \frac{c_k}{12} \sum_{i=1}^L \operatorname*{Res}_{\xi_i=0} (\theta_i'''(\xi_i)\eta_i(\xi_i)\, d\xi_i)\, \mathrm{id}, \tag{1.60}$$

$$[T^*\{\theta\}, T^*\{\eta\}] = T^*\{[\theta, \eta]\} - \frac{c_k}{12} \sum_{i=1}^L \operatorname*{Res}_{\xi_i=0} (\theta_i'''(\xi_i)\eta_i(\xi_i)\, d\xi_i)\, \mathrm{id}, \tag{1.61}$$

where $c_k = k \dim \mathfrak{g}/\kappa$ and $\kappa = k + h^\vee$. Namely, the definitions (1.57) *and* (1.58) *define representations of $\mathcal{V}ir^D$ on $M_k(V)$ with central charge c_k and on $(M_k(V))^*$ with central charge $-c_k$ respectively.*

This is a direct consequence of the definitions (1.57), (1.58) and the Virasoro commutation relations (1.52).

As shown in Lemma 1.11, we have a global meromorphic 2-differential $\Phi(T(t)v)(dt)^2$ for $\Phi \in \mathrm{CB}_k(Q, V)$ and $v \in M_k(V)$. Therefore (1.59) and the residue theorem imply the following lemma.

Lemma 1.14. *Let Φ be a conformal block in $\mathrm{CB}_k(Q, V)$, $\theta(t)\partial_t$ in $H^0(X, \mathcal{T}_X(*D))$, and $\theta_i(\xi_i)\partial_{\xi_i}$ the Laurent expansion of $\theta(t)\partial_t$ in ξ_i for each $i = 1, \dots, L$. Denote $(\theta_i(\xi_i)\partial_{\xi_i})_{i=1}^L$ by θ. Then $T^*\{\theta\}\Phi = 0$.*

2. Critical level and the XYZ Gaudin model

In this section we restrict ourselves to the case $k = -h^\vee = -N$, namely the case when the level is critical.

We showed in §1.2 that the conformal block is determined by its finite-dimensional part. (See Proposition 1.5.) We shall see in this section that the Sugawara tensor is expressed by integrals of motion of the XYZ Gaudin model on this finite-dimensional space. Indeed, it shall be shown that determining certain spaces of conformal blocks at the critical level is equivalent to solving the XYZ Gaudin model.

First let us recall the definition of the XYZ Gaudin model, generalizing the definition in [Ga1], [ST1] to sl_N case. Keeping in mind that we will show the relation of the conformal field theory and the XYZ Gaudin model, we use the same notation for sl_N-modules, points Q_i on a elliptic curve etc. as in the previous section §1, and fix local coordinates at each point Q_i to $\xi_i = t - z_i$, where t is the global coordinate of $\mathbb{C}$.

The Hilbert space of the model is a tensor product of the finite-dimensional irreducible representation spaces of $sl_N(\mathbb{C})$: $V := \bigotimes_{i=1}^L V_i$. The generating function $\hat{\tau}(u)$ of the integrals of motion of the model is defined as the trace of square of the quasi-classical limit $\mathcal{T}(u)$ of the monodromy matrix of the spin chain model

associated with the Baxter-Belavin's elliptic R-matrix:

$$\mathcal{T}(u) := \sum_{i=1}^{L} \sum_{(a,b)\neq(0,0)} w_{a,b}(u-z_i) J_{a,b} \otimes \rho_i(J^{a,b}), \tag{2.1}$$

$$\begin{aligned} \hat{\tau}(u) :=& \frac{1}{2} \operatorname{tr}(\mathcal{T}(u))^2 \\ =& \frac{1}{2} \sum_{i,j=1}^{L} \sum_{(a,b)\neq(0,0)} w_{-a,-b}(u-z_i) w_{a,b}(u-z_j) \rho_i(J_{a,b}) \rho_j(J^{a,b}), \end{aligned} \tag{2.2}$$

where the indices of the summations over (a,b) run through $a = 0, \dots, N-1$, $b = 0, \dots, N-1$, $(a,b) \neq (0,0)$, and $w_{a,b}$ are functions defined by (1.14). As before, ρ_i is the representation of $\mathfrak{g}$ on the i-th factor V_i of V.

The integrals of motion are encoded here in the following way:

$$\hat{\tau}(u) = \sum_{i=1}^{L} C_i \wp(u-z_i) + \sum_{i=1}^{L} H_i \zeta(u-z_i) + H_0, \tag{2.3}$$

where ζ and $\wp$ are Weierstraß' ζ and $\wp$ functions, C_i is the Casimir operator of $\mathfrak{g}$ acting on V_i, i.e.,

$$C_i = \frac{1}{2} \sum_{(a,b)\neq(0,0)} \rho_i(J_{a,b}) \rho_i(J^{a,b}), \tag{2.4}$$

and H_i $(i = 1, \dots, L)$ and H_0 are integrals of motion. Operators H_i satisfy $\sum_{i=1}^{L} H_i = 0$, and hence there are L independent integrals of motion.

Example 2.1. When $N = 2$, the Casimir operator C_i is equal to $l_i(l_i+1)\mathrm{id}_{V_i}$, where $l_i = (\dim V_i - 1)/2$, and H_i are expressed as (cf. [ST1]):

$$\begin{aligned} H_i =& \sum_{j\neq i} \sum_{(a,b)=(0,1),(1,1),(1,0)} w_{a,b}(z_i - z_j) \rho_i(J_{a,b}) \rho_j(J^{a,b}), \\ H_0 =& \frac{1}{2} \sum_{i=1}^{L} \sum_{(a,b)=(0,1),(1,1),(1,0)} \Big(-e_{a,b} \rho_i(J_{a,b}) \rho_i(J^{a,b}) \\ & + \sum_{j\neq i} w_{a,b}(z_i - z_j) \left(\zeta\Big(z_i - z_j + \frac{\omega_{a,b}}{2} \Big) - \zeta\Big(\frac{\omega_{a,b}}{2} \Big) \right) \rho_i(J_{a,b}) \rho_j(J^{a,b}) \Big), \end{aligned} \tag{2.5}$$

where $\omega_{a,b} = a\tau + b$ and $e_{a,b} = \wp(\omega_{a,b}/2)$.

We interpret this system as a twisted WZW model at the critical level. Let us come back to the situation in §1 and put $u = t(P)$. The sl_N-module V_i is assigned to the point Q_i and regarded as $\mathfrak{g}^{\mathrm{tw}}|_{Q_i}$ module by the trivialization (1.35). Assign the vacuum module $M_k(\mathbb{C}_P)$ at P $(k = -h^\vee = -N)$. As before $\tilde{\Phi} \in \mathrm{CB}_k(\{P, Q_i\}, \{\mathbb{C}_P, V_i\})$ corresponds to a conformal block $\Phi \in \mathrm{CB}_k(\{Q_i\}, \{V_i\})$ through the isomorphism (1.31). The correlation function of the Sugawara tensor $\tilde{\Phi}(S(t)(u_P \otimes v))\,(dt)^2$ has an expansion (1.46) at Q_i and at P,

$$\begin{aligned} \tilde{\Phi}(S(t)(u_P \otimes v))\,(dt)^2 &= \sum_{m\in\mathbb{Z}} \tilde{\Phi}(u_P \otimes \rho_i(S[m])v)(t-z_i)^{-m-2}\,(dt)^2 \\ &= \sum_{m\in\mathbb{Z}} \tilde{\Phi}(S[m]u_P \otimes v)(t-u)^{-m-2}\,(dt)^2, \end{aligned} \tag{2.6}$$

respectively. The right-hand side of (2.6) is $\sum_{m\le -2}\tilde\Phi(S[m]u_P\otimes v)(t-u)^{-m-2}(dt)^2$, since $S[m]u_P=0$ for all $m\geqq -1$. Hence, evaluating (2.6) at $t=u$, we have

$$\sum_{m\in\mathbb{Z}}\Phi(\rho_i(S[m])v)(u-z_i)^{-m-2}=\tilde\Phi(S[-2]u_P\otimes v). \tag{2.7}$$

Lemma 2.2. *Let v be a vector in $V=\bigotimes_{i=1}^L V_i$. Then we have*

$$S[-2]u_P\otimes v\equiv u_P\otimes\hat\tau(u)v\quad \text{in } \mathrm{CC}_k(\{P,Q_i\},\{\mathbb{C}_P,V_i\}).$$

Hence the right-hand side of (2.7) *is equal to* $\Phi(\hat\tau(u)v)$.

Proof. First note that

$$S[-2]u_P=\frac12\sum_{(a,b)\neq(0,0)}J_{a,b}[-1]J^{a,b}[-1]u_P.$$

The key step is to exchange the operators $J_{a,b}[-1]$ and $J^{a,b}[-1]$ with operators acting on v by using the Ward identity (1.24). Recall that the functions $w_{a,b}(t-u)$ (1.14) in $K_{a,b}$ define meromorphic sections

$$J_{a,b,P}(t):=w_{a,b}(t-u)J_{a,b},\quad J^{a,b,P}(t):=w_{-a,-b}(t-u)J^{a,b}, \tag{2.8}$$

of $\mathfrak{g}^{\mathrm{tw}}$ through the inclusion (1.13) and (1.10). These sections belong to $\pi_*\mathfrak{g}^{\mathrm{tw}}(P)$. Since $J_{a,b,P}(t)$ has a Laurent expansion

$$J_{a,b,P}(t)=\frac{J_{a,b}}{t-u}+w_{a,b,0}J_{a,b}+w_{a,b,1}J_{a,b}\cdot(t-u)+O((t-u)^2) \tag{2.9}$$

at P (see (1.15)), and $J^{a,b}(t)$ has a similar expansion, we have

$$J_{a,b}[-1]J^{a,b}[-1]u_P=(J_{a,b,P}(t))_P(J^{a,b,P}(t))_Pu_P-kw_{a,b,1}u_P, \tag{2.10}$$

where $k=-h^\vee=-N$. Summing up (2.10) for (a,b) and using (1.18), we obtain

$$S[-2]u_P=\frac12\sum_{(a,b)\neq(0,0)}(J_{a,b,P}(t))_P\,(J^{a,b,P}(t))_Pu_P. \tag{2.11}$$

Substituting (2.11) into $S[-2]u_P\otimes v$ and swapping $J_{a,b,P}(t)$ and then $J^{a,b,P}(t)$ by the Ward identity (1.24), we obtain

$$S[-2]u_P\otimes v\equiv u_P\otimes\frac12\sum_{i,j=1}^L\sum_{(a,b)\neq(0,0)}w_{a,b}(z_i-u)w_{-a,-b}(z_j-u)\rho_i(J_{a,b})\rho_j(J^{a,b})v, \tag{2.12}$$

which proves the lemma because of (1.17). □

Corollary 2.3. $[\hat\tau(u),\hat\tau(u')]=0$ *for any u and u'. In particular, H_i $(i=0,1,\dots,L)$ commute with each other.*

Proof. Since the Sugawara operators $S[m]$ commute with the affine Lie algebra at the critical level due to (1.48), we have

$$A[n]S[m]u_P=0\quad\text{for } A\in\mathfrak{g} \text{ and } n\geqq 0.$$

Hence we can find the following formula in the similar way as the proof of Lemma 2.2:

$$S[-2]u_{P'}\otimes S[m]u_P\otimes v\equiv u_{P'}\otimes S[m]u_P\otimes\hat\tau(u')v,$$

where $t(P) = u$, $t(P') = u'$, and $v \in V$. Using this formula and Lemma 2.2, we obtain

$$u_{P'} \otimes u_P \otimes \hat\tau(u)\hat\tau(u')v \equiv u_{P'} \otimes S[-2]u_P \otimes \hat\tau(u')v \equiv S[-2]u_{P'} \otimes S[-2]u_P \otimes v$$
$$\equiv S[-2]u_{P'} \otimes u_P \otimes \hat\tau(u)v \equiv u_{P'} \otimes u_P \otimes \hat\tau(u')\hat\tau(u)v.$$

This proves the corollary in view of Proposition 1.5. □

Once the correspondence of the Hamiltonians of the XYZ Gaudin model and the correlation functions of the twisted WZW model is established, the eigenvalue problem of the XYZ Gaudin model is rewritten in terms of the conformal block of the twisted WZW model, as is the case with the (XXX) Gaudin model. (See [Fr2].) We sketch below how it goes, restricting ourselves to sl_2 case. For general sl_N case, we should introduce higher order Sugawara operators, whose constructions are found in [Hay] and [GW].

Let us introduce a meromorphic (single-valued) function on X of the form

$$q(t) = \sum_{i=1}^{L} l_i(l_i+1)\wp(t-z_i) + \sum_{i=1}^{L} \mu_i \zeta(t-z_i) + \mu_0, \tag{2.13}$$

where $l_i = (\dim V_i - 1)/2$ (cf. Example 2.1), μ_i and μ_0 are parameters satisfying $\sum_{i=1}^{L} \mu_i = 0$. Let $q_i(t-z_i) = \sum_{n\in\mathbb{Z}} q_{i,n}(t-z_i)^{-n-2}$ be the Laurent expansion of $q(t)$ at Q_i. Denote by $K^{q_i}(V_i)$ the submodule of $M_{-2}(V_i)$ generated by the vectors $(S[m] - q_{i,m})v_i$ for $v_i \in V_i$, $m \in \mathbb{Z}$ and put $M^{q_i}(V_i) := M_{-2}(V_i)/K^{q_i}(V_i)$.

Theorem 2.4. *The space of conformal coinvariants and that of conformal blocks associated to the module $M^q(V) := \bigotimes_{i=1}^{L} M^{q_i}(V_i)$ are isomorphic to the quotient of $V := \bigotimes_{i=1}^{L} V_i$ by the subspace $J^\mu(V)$ spanned by vectors of the form $(H_i - \mu_i)v$ for $i = 0, 1, \dots, L$ and $v \in V$ and its dual:*

$$\mathrm{CC}_k(M^q(V)) \cong V/J^\mu(V), \quad \mathrm{CB}_k(M^q(V)) \cong (V/J^\mu(V))^*.$$

Proof. We prove the statement for the conformal blocks. The statement for the space of conformal coinvariants follows from this since it is finite-dimensional and dual to the space of conformal blocks.

Let Φ be any linear functional on $M_{-2}(V) = \bigotimes_{i=1}^{L} M_{-2}(V_i)$. A necessary and sufficient condition for Φ to be a conformal block in $\mathrm{CB}_k(M^q(V))$ is that it vanishes on $\mathfrak{g}_X^D M_{-2}(V)$ and on the subspaces

$$K^q(V) := \sum_{i=1}^{L} M_{-2}(V_1) \otimes \cdots \otimes K^{q_i}(V_i) \otimes \cdots \otimes M_{-2}(V_L).$$

First we show that this condition implies $\Phi((H_i - \mu_i)v) = 0$ for $i = 0, 1, \dots, L$ and $v \in V$.

The assumption is encapsulated in the following expression by a generating function,

$$\sum_{m\in\mathbb{Z}} \Phi(\rho_i(S[m] - q_{i,m})v)(u - z_i)^{-m-2} = 0, \tag{2.14}$$

which means $\Phi(\hat\tau(u)v) = q(u)\Phi(v)$ because of (2.7) and Lemma 2.2. Thus (2.3) and (2.13) shows that $\Phi(H_i v) = \mu_i \Phi(v)$.

We prove the converse statement next. Assume that Φ vanishes on the subspace $J^\mu(V)$. Let v be an arbitrary vector in $M_{-2}(V)$. We want to show that $\Phi(\rho_i(S[m]-$

$q_{i,m})v)$ vanishes for any m and i, but for this purpose we may assume $v \in V$ without loss of generality. Indeed any v can be written in the form $v = g_{\dot{X}} v^0$ by the decomposition (1.29), where $g_{\dot{X}} \in U(\mathfrak{g}^D_{\dot{X}})$, $v^0 \in V$, and therefore

$$\rho_i(S[m] - q_{i,m})v = g_{\dot{X}} \rho_i(S[m] - q_{i,m})v^0,$$

since $S[m]$ belongs to the center of $U_{-2}(\widehat{sl}(2))$. The Ward identity (1.25) implies that $\Phi(\rho_i(S[m] - q_{i,m})v) = 0$ if $\Phi(\rho_i(S[m] - q_{i,m})v^0) = 0$.

For $v \in V$, we can prove $\Phi(\rho_i(S[m] - q_{i,m})v) = 0$ by tracing back the first part of this proof. $\square$

3. Sheaves of conformal coinvariants and conformal blocks

So far we have fixed the modulus τ of an elliptic curve and marked points on it. In this section we introduce sheaves of conformal coinvariants and conformal blocks on a family of pointed elliptic curves.

3.1. **Family of pointed elliptic curves and Lie algebra bundles.** In this subsection we construct a family of elliptic curves with marked points, a group bundle, and the associated Lie algebra bundle over this family. The fiber at a point of the base space of the family gives the group bundle G^{tw} and the Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}$ on a pointed elliptic curve defined in §1.1.

Recall that $\mathfrak{H}$ denotes the upper half plane. We define $\widetilde{\mathfrak{X}}$ and S by

$$S := \{ (\tau; z) = (\tau; z_1, \dots, z_L) \in \mathfrak{H} \times \mathbb{C}^L \mid z_i - z_j \notin \mathbb{Z} + \tau\mathbb{Z} \text{ if } i \neq j \},$$
$$\widetilde{\mathfrak{X}} := S \times \mathbb{C}.$$

Let $\tilde{\pi} = \pi_{\widetilde{\mathfrak{X}}/S}$ be the projection from $\widetilde{\mathfrak{X}}$ onto S along $\mathbb{C}$ and $\tilde{q}_i$ the section of $\tilde{\pi}$ given by z_i:

$$\tilde{q}_i(\tau; z) := (\tau; z; z_i) \in \widetilde{\mathfrak{X}} \quad \text{for } (\tau; z) = (\tau; z_1, \dots, z_L) \in S.$$

A *family of L-pointed elliptic curves* $\pi : \mathfrak{X} \twoheadrightarrow S$ is constructed as follows. Define the action of $\mathbb{Z}^2$ on $\widetilde{\mathfrak{X}}$ by

$$(m, n) \cdot (\tau; z; t) := (\tau; z; t + m\tau + n) \quad \text{for } (m, n) \in \mathbb{Z}^2,\ (\tau; z; t) \in \widetilde{\mathfrak{X}}. \tag{3.1}$$

Let $\mathfrak{X}$ be the quotient space of $\widetilde{\mathfrak{X}}$ by the action of $\mathbb{Z}^2$:

$$\mathfrak{X} := \mathbb{Z}^2 \backslash \widetilde{\mathfrak{X}}. \tag{3.2}$$

Let $\pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}}$ be the natural projection from $\widetilde{\mathfrak{X}}$ onto $\mathfrak{X}$ and $\pi = \pi_{\mathfrak{X}/S}$ the projection from $\mathfrak{X}$ onto S induced by $\tilde{\pi}$. We put

$$q_i := \pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}} \circ \tilde{q}_i, \quad Q_i := q_i(S), \quad D := \bigcup_{i=1}^L Q_i, \quad \dot{\mathfrak{X}} := \mathfrak{X} \setminus D.$$

Here q_i is the section of π induced by $\tilde{q}_i$ and D is also regarded as a divisor $\sum_{i=1}^L Q_i$ on $\mathfrak{X}$. The fiber of π at $(\tau, z) = (\tau; z_1, \dots, z_L) \in S$ is an elliptic curve with modulus τ and marked points $z_1, \dots, z_L$.

A group bundle $G^{\mathrm{tw}}_{\mathfrak{X}}$ and a Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ on $\mathfrak{X}$ are defined as follows. Due to the definition of $\mathfrak{X}$, the Galois group of the covering $\pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}} : \widetilde{\mathfrak{X}} \twoheadrightarrow \mathfrak{X}$ is naturally identified with $\mathbb{Z}^2$. Its natural right action on $\widetilde{\mathfrak{X}}$ is given by $(\tau; z; t) \cdot (m, n) :=$

$(-m,-n)\cdot(\tau;z;t)$. Then the covering $\pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}} : \widetilde{\mathfrak{X}} \twoheadrightarrow \mathfrak{X}$ is regarded as a principal $\mathbb{Z}^2$-bundle on $\mathfrak{X}$. The actions of the Galois group $\mathbb{Z}^2$ on G and $\mathfrak{g}$ are defined by

$$(m,n)\cdot g := (\beta^m\alpha^n)g(\beta^m\alpha^n)^{-1} \quad \text{for } g\in G \text{ and } (m,n)\in\mathbb{Z}^2, \tag{3.3}$$

$$(m,n)\cdot A := (\beta^m\alpha^n)A(\beta^m\alpha^n)^{-1} \quad \text{for } A\in \mathfrak{g} \text{ and } (m,n)\in\mathbb{Z}^2. \tag{3.4}$$

These actions produces the associated group bundle $G^{\mathrm{tw}}_{\mathfrak{X}}$ and the associated Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ on $\mathfrak{X}$:

$$G^{\mathrm{tw}}_{\mathfrak{X}} := \widetilde{\mathfrak{X}} \times^{\mathbb{Z}^2} G, \quad \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} := \widetilde{\mathfrak{X}} \times^{\mathbb{Z}^2} \mathfrak{g}. \tag{3.5}$$

Their fibers at a point $(\tau;z)\in S$ can be identified with G^{tw} and $\mathfrak{g}^{\mathrm{tw}}$ in §1.1.

We denote the $\mathcal{O}_{\mathfrak{X}}$-Lie algebra associated to the Lie algebra bundle $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ by the same symbol $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$, as mentioned in §1.2. The sheaf $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ can be written in the following form:

$$\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} = \{\, A \in (\pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}})_*(\mathfrak{g}\otimes\mathcal{O}_{\widetilde{\mathfrak{X}}}) \mid A(p\cdot\tilde{x}) = p\cdot A(\tilde{x}) \text{ for } \tilde{x}\in\widetilde{\mathfrak{X}},\ p\in\mathbb{Z}^2 \,\}. \tag{3.6}$$

Hence if we take an open subset U' of $\widetilde{\mathfrak{X}}$ which does not intersect $(m,n)\cdot U'$ for any $(m,n)\in\mathbb{Z}^2\smallsetminus\{(0,0)\}$ and denote by U the image of U' on $\mathfrak{X}$, then the restriction of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ on U can be canonically identified with $\mathfrak{g}\otimes\mathcal{O}_U$:

$$\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}|_U \cong (\pi_{U'/U})_*(\mathfrak{g}\otimes\mathcal{O}_{U'}) = \mathfrak{g}\otimes\mathcal{O}_U, \tag{3.7}$$

where $\pi_{U'/U}$ is the natural biholomorphic projection $U' \xrightarrow{\sim} U$.

Denote by $\Omega^1_{\mathfrak{X}}$ the sheaf of 1-forms on $\mathfrak{X}$ and by $\Omega^1_{\mathfrak{X}/S}$ the sheaf of relative differentials on $\mathfrak{X}$ over S. It follows from the definition of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ that $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ possesses a natural connection $\nabla : \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} \to \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}}$, which is induced by the trivial connection $\mathrm{id}\otimes d : \mathfrak{g}\otimes\mathcal{O}_{\widetilde{\mathfrak{X}}} \to \mathfrak{g}\otimes\Omega^1_{\widetilde{\mathfrak{X}}}$ through the identification (3.6). This means that, under the trivialization (3.7), the connection ∇ is identified with $\mathrm{id}\otimes d_U$ where d_U is the exterior derivation on U. The relative connection $\nabla_{\mathfrak{X}/S}$ along the fibers is defined to be the composite of the connection ∇ and the natural homomorphism $\Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{X}/S}$. Under the trivialization (3.7) and the coordinate $(\tau;z;t)$, the relative connection $\nabla_{\widetilde{\mathfrak{X}}/S}$ is equal to the exterior derivation by t.

Define the invariant $\mathcal{O}_X$-inner product on $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ by

$$(A|B) := \frac{1}{2h^\vee}\operatorname{tr}_{\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}}(\operatorname{ad}A\operatorname{ad}B) = \frac{1}{2N}\operatorname{tr}_{\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}}(\operatorname{ad}A\operatorname{ad}B) \in \mathcal{O}_X \quad \text{for } A,B\in\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}, \tag{3.8}$$

where the symbol ad denotes the adjoint representation of the $\mathcal{O}_{\mathfrak{X}}$-Lie algebra $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$. Under the trivialization (3.7), the inner product on $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ is equal to the inner product defined by (1.1) and hence it is invariant under the translation along the connection ∇.

Recall that $\varepsilon = \exp(2\pi i/N)$. For $(a,b)\in(\mathbb{Z}/N\mathbb{Z})^2$, the 1-dimensional representation $(m,n)\mapsto\varepsilon^{bm+an}$ of $\mathbb{Z}^2$ defines the associated flat line bundle $L_{a,b}$ on $\mathfrak{X}$. We obtain the decomposition of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ into line bundles:

$$\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} = \bigoplus_{(a,b)\neq(0,0)} J_{a,b}L_{a,b}. \tag{3.9}$$

This is a sheaf version of (1.7).

Lemma 3.1. *$R^p\pi_*\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} = 0$ for all p.*

Proof. Since $L_{a,b}^* \otimes_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/S}$ is isomorphic to $L_{-a,-b}$, it follows that $R^1\pi_* L_{a,b} \cong \mathcal{H}om_{\mathcal{O}_S}(\pi_* L_{-a,-b}, \mathcal{O}_S)$ by the Serre duality. Therefore, because of the decomposition (3.9), it is enough to show $\pi_* L_{a,b} = 0$ for $(a,b) \neq (0,0)$. Let U be any open subset of S and put $V := \pi^{-1}(U)$. For each $s = (\tau; z) \in U$, the restriction $L_{a,b}|_{X_s}$ of $L_{a,b}$ on the fiber $X_s := \pi^{-1}(s)$ can be identified with the line bundle $L_{a,b}$ on X_τ define in §1.1. Hence we obtain $H^0(X_s, L_{a,b}|_{X_s}) = 0$ for each $s \in S$ In particular, for every $f \in H^0(V, L_{a,b}) = H^0(U, \pi_* L_{a,b})$, the restriction $f|_{X_s}$ of f on the fiber vanishes for each $s \in S$ and hence f vanishes itself. This means that $H^0(U, \pi_* L_{a,b}) = 0$. We have proved the lemma. □

3.2. **Sheaf of affine Lie algebras.** In this section we define a sheaf version of the Lie algebras $\hat{\mathfrak{g}}^D$, $\mathfrak{g}^D_{\dot{X}}$, etc. on the base space S of the family.

For an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ and a closed analytic subset W of $\mathfrak{X}$, the restriction $\mathcal{F}|_W$ of $\mathcal{F}$ on W and the completion $\hat{\mathcal{F}}_{|W} = (\mathcal{F})^\wedge_W$ of $\mathcal{F}$ at W are defined by

$$\mathcal{F}|_W := \mathcal{F}/I_W\mathcal{F}, \quad \widehat{\mathcal{F}}_{|W} = (\mathcal{F})^\wedge_W := \operatorname*{proj\,lim}_{n\to\infty}(\mathcal{F}/I_W^n\mathcal{F}), \tag{3.10}$$

where I_W is the defining ideal of W in $\mathfrak{X}$.

We define the $\mathcal{O}_S$-Lie algebras $\mathfrak{g}_S^{Q_i}$, $\mathfrak{g}_{S,+}^{Q_i}$, $\mathfrak{g}_S^D$, $\mathfrak{g}_{S,+}^D$, and $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$ as follows:

$$\begin{aligned}
&\mathfrak{g}_S^{Q_i} := \pi_*(\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*Q_i))^\wedge_{Q_i}, && \mathfrak{g}_{S,+}^{Q_i} := \pi_*(\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}})^\wedge_{Q_i}, \\
&\mathfrak{g}_S^D := \pi_*(\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*D))^\wedge_D = \bigoplus_{i=1}^L \mathfrak{g}_S^{Q_i}, && \mathfrak{g}_{S,+}^D := \pi_*(\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}})^\wedge_D = \bigoplus_{i=1}^L \mathfrak{g}_{S,+}^{Q_i}, \\
&\mathfrak{g}_{\dot{\mathfrak{X}}}^D := \pi_*(\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*D)).
\end{aligned} \tag{3.11}$$

The 2-cocycle of $\mathfrak{g}_S^D$ is defined by

$$\mathrm{c_a}(A,B) := \sum_{i=1}^L \mathrm{Res}_{Q_i}(\nabla A_i|B_i) = \sum_{i=1}^L \mathrm{Res}_{Q_i}(\nabla_{\mathfrak{X}/S} A_i|B_i), \tag{3.12}$$

where $A = (A_i)_{i=1}^L, B = (B_i)_{i=1}^L \in \mathfrak{g}_S^D$ and Res_{Q_i} is the residue along Q_i. Using the 2-cocycle $\mathrm{c_a}(\cdot,\cdot)$, we define a central extension $\hat{\mathfrak{g}}_S^D$ of $\mathfrak{g}_S^D$:

$$\hat{\mathfrak{g}}_S^D := \mathfrak{g}_S^D \oplus \mathcal{O}_S \hat{k}, \tag{3.13}$$

where its Lie algebra structure is defined by the formula similar to (1.22). Put $\hat{\mathfrak{g}}_S^{Q_i} := \mathfrak{g}_S^{Q_i} \oplus \mathcal{O}_S\hat{k}$, which is a $\mathcal{O}_S$-Lie subalgebra of $\hat{\mathfrak{g}}_S^D$. We call $\hat{\mathfrak{g}}_S^D$ (resp. $\hat{\mathfrak{g}}_S^{Q_i}$) the *sheaf of affine Lie algebras at D* (resp. Q_i).

The diagonal embedding of $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$ into $\mathfrak{g}_S^D$ is defined to be the mapping which sends $A \in \mathfrak{g}_{\dot{\mathfrak{X}}}^D$ to $(A_i)_{i=1}^L \in \mathfrak{g}_S^D$, where each A_i is the image of A given by the natural embedding $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*D) \hookrightarrow (\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*Q_i))^\wedge_{Q_i}$. We identify $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$ with its image in $\mathfrak{g}_S^D$ and $\hat{\mathfrak{g}}_S^D$. For $A, B \in \mathfrak{g}_{\dot{\mathfrak{X}}}^D$, we can regard $(\nabla_{\mathfrak{X}/S}A|B)$ as an element of $\pi_*\Omega^1_{\mathfrak{X}/S}(*D)$. Hence, using the residue theorem, we obtain that $\mathrm{c_a}(A,B) = 0$. Thus $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$, as well as $\mathfrak{g}_{S,+}^{Q_i}$, is an $\mathcal{O}_S$-Lie subalgebra of $\hat{\mathfrak{g}}_S^D$.

Put $\hat{\mathfrak{g}}_{S,+}^D := \mathfrak{g}_S^D \oplus \mathcal{O}_S\hat{k}$. Then Lemma 3.1 implies the sheaf version of (1.29).

Lemma 3.2. $\hat{\mathfrak{g}}_S^D = \mathfrak{g}_{\dot{\mathfrak{X}}}^D \oplus \hat{\mathfrak{g}}_{S,+}^D$.

Proof. We can calculate $R^p\pi_*\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}$ for $p = 0, 1$ as the kernel and the cokernel of $\mathfrak{g}_{\dot{\mathfrak{X}}}^D \oplus \mathfrak{g}_{S,+}^D \to \mathfrak{g}_S^D$, which sends $(a_{\dot{\mathfrak{X}}}; a_+)$ to $a_{\dot{\mathfrak{X}}} - a_+$. But then Lemma 3.1 means

that both the kernel and the cokernel vanish and hence $\mathfrak{g}^D_{\dot{\mathfrak{X}}} \oplus \mathfrak{g}^D_{S,+} = \mathfrak{g}^D_S$. We have proved the lemma. □

Choose any open neighborhood U' of $\widetilde{Q}_i := \tilde{q}_i(S)$ which does not intersect $(m,n)\cdot U'$ for any $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then, applying the trivialization (3.7) to U', we obtain a natural isomorphism

$$\mathfrak{g}^{Q_i}_S \cong \mathfrak{g} \otimes \pi_*(\widehat{\mathcal{O}}_{\mathfrak{X}|Q_i}(*Q_i)), \tag{3.14}$$

which does not depend on the choice of U' and is defined globally on S. Furthermore, using the coordinate $(\tau; z; \xi_i)$ with $\xi_i = t - z_i$, we have the following isomorphism defined over S:

$$\mathfrak{g}^{Q_i}_S \cong \mathfrak{g} \otimes \mathcal{O}_S((\xi_i)). \tag{3.15}$$

Under this trivialization, $\mathfrak{g}^{Q_i}_{S,+}$ is identified with $\mathfrak{g} \otimes \mathcal{O}_S[[\xi_i]]$ and the connections on $\mathfrak{g}^{Q_i}_S$ induced by ∇ and $\nabla_{\mathfrak{X}/S}$ are written in the following forms:

$$\nabla A = \frac{\partial A}{\partial \tau}\, d\tau + \sum_{i=1}^{L} \frac{\partial A}{\partial z_i}\, dz_i + \frac{\partial A}{\partial \xi_i}\, d\xi_i \in \mathfrak{g} \otimes \Omega^1_S((\xi_i)) \oplus \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))\, d\xi_i, \tag{3.16}$$

$$\nabla_{\mathfrak{X}/S} A = \frac{\partial A}{\partial \xi_i}\, d\xi_i \in \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))\, d\xi_i, \tag{3.17}$$

where $A \in \mathfrak{g} \otimes \mathcal{O}_S((\xi_i)) \cong \mathfrak{g}^{Q_i}_S$. We also obtain the induced global trivialization of the sheaf of affine Lie algebras on S:

$$\hat{\mathfrak{g}}^D_S \cong \bigoplus_{i-1}^{L} \mathfrak{g} \otimes \mathcal{O}_S((\xi_i)) \oplus \mathcal{O}_S \hat{k}. \tag{3.18}$$

Under this trivialization, the bracket of $\hat{\mathfrak{g}}^D_S$ is represented in the following form:

$$\big[(A_i \otimes f_i)_{i=1}^L, (B_i \otimes g_i)_{i=1}^L\big] = \big([A_i, B_i] \otimes f_i g_i\big)_{i=1}^L + \hat{k} \sum_{i=1}^{L} (A_i|B_i) \operatorname*{Res}_{\xi_i=0}(df_i \cdot g), \tag{3.19}$$

where $A_i, B_i \in \mathfrak{g}$ and $f_i, g_i \in \mathcal{O}_S((\xi_i))$.

3.3. **Definition of the sheaves of conformal coinvariants and conformal blocks.** For any $\mathcal{O}_S$-Lie algebra $\mathfrak{a} = \mathfrak{g}^D_{S,+}, \hat{\mathfrak{g}}^D_S$, etc., we denote by $U_S(\mathfrak{a})$ the universal $\mathcal{O}_S$-enveloping algebra of $\mathfrak{a}$ and define the category of $\mathfrak{a}$-modules to be that of $U_S(\mathfrak{a})$-modules.

Definition 3.3. For any $\hat{\mathfrak{g}}^D_S$-module $\mathcal{M}$, we define the *sheaf* $\mathcal{CC}(\mathcal{M})$ *of conformal coinvariants* and the *sheaf* $\mathcal{CB}(\mathcal{M})$ *of conformal blocks* by

$$\mathcal{CC}(\mathcal{M}) := \mathcal{M}/\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}, \tag{3.20}$$

$$\mathcal{CB}(\mathcal{M}) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{CC}(\mathcal{M}), \mathcal{O}_S). \tag{3.21}$$

Namely, the $\mathcal{O}_S$-module $\mathcal{CC}(\mathcal{M})$ is generated by $\mathcal{M}$ with relations

$$A_{\dot{\mathfrak{X}}} v \equiv 0 \tag{3.22}$$

for all $A_{\dot{\mathfrak{X}}} \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$, $v \in \mathcal{M}$, and $\Phi \in \mathcal{CB}(\mathcal{M})$ means that Φ belongs to $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \mathcal{O}_S)$ and satisfies

$$\Phi(A_{\dot{\mathfrak{X}}} v) = 0 \tag{3.23}$$

for all $A_{\dot{\mathfrak{X}}} \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$, $v \in \mathcal{M}$. These equations, (3.22) and (3.23), are also called the *Ward identities*. We can regard $\mathcal{CC}(\cdot)$ as a covariant right exact functor from the category of $\hat{\mathfrak{g}}^D_S$-modules to that of $\mathcal{O}_S$-modules and similarly $\mathcal{CB}(\cdot)$ as a contravariant left exact functor.

The $\hat{\mathfrak{g}}^D_S$-modules of our concern are the sheaf version $\mathcal{M}_k(V)$ of $M_k(V)$ in §1.2. We give two equivalent definitions of $\mathcal{M}_k(V)$.

First definition of $\mathcal{M}_k(V)$. Fix an arbitrary complex number k. For each $i = 1, \dots, L$, let M_i be a representation with level k of the affine Lie algebra $\hat{\mathfrak{g}}_i := \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}\hat{k}$. (Here $\hat{\mathfrak{g}}_i$-modules are said to be of level k if the canonical central element $\hat{k}$ acts on them as $k \cdot \mathrm{id}$.) Assume the *smoothness* of M_i, namely, for any $v_i \in M_i$, there exists $m \geqq 0$ such that, for $A_{i_1}, \dots, A_{i_\nu} \in \mathfrak{g}$, $m_1, \dots, m_\nu \geqq 0$, and $\nu = 0, 1, 2, \dots$,

$$(A_{i_1} \otimes \xi_i^{m_1} \mathcal{O}_S[[\xi_i]]) \cdots (A_{i_\nu} \otimes \xi_i^{m_\nu} \mathcal{O}_S[[\xi_i]]) v_i = 0 \quad \text{if } m_1 + \cdots + m_\nu \geqq m. \tag{3.24}$$

Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Then M is a representation with level k of the affine Lie algebra $(\mathfrak{g}^{\oplus L})^\wedge := \bigoplus_{i=1}^L \bigl(\mathfrak{g} \otimes \mathbb{C}((\xi_i))\bigr) \oplus \mathbb{C}\hat{k}$ associated to $\mathfrak{g}^{\oplus L}$. We can define the $\hat{\mathfrak{g}}^D_S$-module structure on $\mathcal{M}$ by

$$(A_i \otimes f_i(\xi_i))_{i=1}^L (v \otimes a) := \sum_{i=1}^L \sum_{m \in \mathbb{Z}} (\rho_i(A_i \otimes \xi_i^m) v) \otimes (f_{i,m} a), \quad \hat{k} v := k v, \tag{3.25}$$

where $A_i \in \mathfrak{g}$, $f_i(\xi_i) = \sum_m f_{i,m} \xi_i^m \in \mathcal{O}_S((\xi_i))$, $f_{i,m}, a \in \mathcal{O}_S$, $v \in M$, and $\rho_i(A_i \otimes \xi_i^m)$ denotes the action of $A_i \otimes \xi_i^m$ on the i-th factors in v. If each M_i is the Weyl module $M_k(V_i)$ induced up from a finite-dimensional irreducible representation V_i of $\mathfrak{g}$, then we put $V := \bigotimes_{i=1}^L V_i$, $M := M_k(V) := \bigotimes_{i=1}^L M_k(V_i)$, and $\mathcal{M} := \mathcal{M}_k(V) := M_k(V) \otimes \mathcal{O}_S$ and denote $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ by $\mathcal{CC}_k(V)$ and $\mathcal{CB}_k(V)$ respectively.

Second definition of $\mathcal{M}_k(V)$. Let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ and put $V := \bigotimes_{i=1}^L V_i$. Denote the constant sheaf associated to V by the same symbol V. Using the trivialization (3.18), we can define the action of $\hat{\mathfrak{g}}_{S,+} = \mathfrak{g}^D_{S,+} \oplus \mathcal{O}_S \hat{k}$ on $V \otimes \mathcal{O}_S$ by

$$(A_i \otimes f_i(\xi_i))_{i=1}^L (v \otimes a) := \sum_i (\rho_i(A_i) v) \otimes (f_i(0) a), \quad \hat{k} v := k v \tag{3.26}$$

where $A_i \in \mathfrak{g}$, $f_i(\xi_i) \in \mathcal{O}_S[[\xi_i]]$, $a \in \mathcal{O}_S$, $v \in V$, and $\rho_i(A_i)$ is the action of A_i on the i-th factors in v. The $\hat{\mathfrak{g}}^D_S$-module $M_{S,k}(V)$ induced from $V \otimes \mathcal{O}_S$ is defined by

$$M_{S,k}(V) := \mathrm{Ind}_{\hat{\mathfrak{g}}^D_{S,+}}^{\hat{\mathfrak{g}}^D_S} (V \otimes \mathcal{O}_S) = U_S(\hat{\mathfrak{g}}^D_S) \otimes_{U_S(\hat{\mathfrak{g}}^D_{S,+})} (V \otimes \mathcal{O}_S) \tag{3.27}$$

Using the decomposition

$$\hat{\mathfrak{g}}^D_S = \left(\bigoplus_{i=1}^L \mathfrak{g} \otimes \xi_i^{-1} \mathcal{O}_S[\xi_i^{-1}] \right) + \hat{\mathfrak{g}}^D_{S,+},$$

we can show that $M_{S,k}(V)$ has the following $\mathcal{O}_S$-free basis:

$$\rho_{i_1}(A_{s_1}[m_1]) \cdots \rho_{i_\nu}(A_{s_\nu}[m_\nu]) v_j, \tag{3.28}$$

where $\nu = 0, 1, 2, \dots,$ $i_n = 1, \dots, L$, $\{A_s\}$ is a basis of $\mathfrak{g}$, $\{v_j\}$ is a basis of V, and $m_1 \leqq \cdots \leqq m_\nu < 0$. This is also an $\mathcal{O}_S$-free basis of $\mathcal{M}_k(V)$ and hence $M_{S,k}(V)$ is isomorphic to $\mathcal{M}_k(V)$ as a $\hat{\mathfrak{g}}^D_S$-module. In the following we identify $\mathcal{M}_k(V)$ with $M_{S,k}(V)$.

This identification of the two definitions and Lemma 3.2 prove the sheaf version of Proposition 1.5.

Proposition 3.4. *Let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ for each i and put $V := \bigotimes_{i=1}^L V_i$. Then the natural inclusion $V \otimes \mathcal{O}_S \hookrightarrow \mathcal{M}_k(V)$ induces the following isomorphisms:*

$$\mathcal{CC}_k(V) \xleftarrow{\sim} V \otimes \mathcal{O}_S \quad \textit{and} \quad \mathcal{CB}_k(V) \xrightarrow{\sim} V^* \otimes \mathcal{O}_S.$$

Proof. From the second definitions of $\mathcal{M}_k(V)$ and Lemma 3.2, it follows that

$$\mathcal{M}_k(V) = U_S(\hat{\mathfrak{g}}^D_S) \otimes_{U_S(\hat{\mathfrak{g}}^D_{S,+})} (V \otimes \mathcal{O}_S) \xleftarrow{\sim} U_S(\mathfrak{g}^D_{\dot{\mathfrak{X}}}) \otimes_{\mathcal{O}_S} (V \otimes \mathcal{O}_S).$$

Namely, $\mathcal{M}_k(V)$ is freely generated by $V \otimes \mathcal{O}_S$ over $U_S(\mathfrak{g}^D_{\dot{\mathfrak{X}}})$. Hence we obtain the formulae

$$\begin{aligned}
\mathcal{CC}_k(V) &= \mathcal{M}_k(V)/\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}_k(V) \xleftarrow{\sim} \left(U_S(\mathfrak{g}^D_{\dot{\mathfrak{X}}})/\mathfrak{g}^D_{\dot{\mathfrak{X}}}U_S(\mathfrak{g}^D_{\dot{\mathfrak{X}}})\right) \otimes_{\mathcal{O}_S} (V \otimes \mathcal{O}_S) \\
&\xleftarrow{\sim} \mathcal{O}_S \otimes_{\mathcal{O}_S} (V \otimes \mathcal{O}_S) = V \otimes \mathcal{O}_S, \\
\mathcal{CB}_k(V) &= \mathcal{H}om_{\mathcal{O}_S}(\mathcal{CC}_k(V), \mathcal{O}_S) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(V \otimes \mathcal{O}_S, \mathcal{O}_S) = V^* \otimes \mathcal{O}_S.
\end{aligned}$$

We have completed the proof of the proposition. □

Corollary 3.5. *For each $i = 1, \dots, L$, let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ and M_i a quotient module of the generalized Verma module $M_k(V_i)$ of the affine Lie algebra $\hat{\mathfrak{g}}_i$. Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Then the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks are $\mathcal{O}_S$-coherent.*

Proof. Since $\mathcal{CB}(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{CC}(\mathcal{M}), \mathcal{O}_S)$, it suffices for the proof to see that $\mathcal{CC}(\mathcal{M})$ is coherent. The right exactness of the functor $\mathcal{CC}(\cdot)$ and the fact that $\mathcal{M}$ is a quotient $\hat{\mathfrak{g}}^D_S$-module of $\mathcal{M}_k(V)$ imply that $\mathcal{CC}(\mathcal{M})$ is a quotient $\mathcal{O}_S$-module of $\mathcal{CC}_k(V) = \mathcal{CC}(\mathcal{M}_k(V))$, which is $\mathcal{O}_S$-coherent due to Proposition 3.4. Hence $\mathcal{CC}(\mathcal{M})$ is also $\mathcal{O}_S$-coherent. □

4. Sheaf of the Virasoro algebras

This section provides the sheaf version of the Virasoro algebras and its actions on representations of the sheaf of affine Lie algebra, which will be used in §5 to endow the sheaf of conformal coinvariants and the sheaf of conformal blocks with $\mathcal{D}_S$-module structures, when the level is not critical (i.e., $\kappa = k + h^\vee \neq 0$).

4.1. **Definition of the sheaf of the Virasoro algebras.** We define the *sheaf of the Virasoro algebras* by

$$\mathcal{V}ir^D_S := \mathcal{T}_S \oplus \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i} \oplus \mathcal{O}_S\hat{c}, \tag{4.1}$$

where $\mathcal{T}_S$ is the tangent sheaf of S. The Lie algebra structure which we shall give to this $\mathcal{O}_S$-sheaf below reduces to the Virasoro algebra structure on $\mathcal{V}ir^D$ (1.55), when S is replaced with a point.

In order to define a Lie algebra structure on $\mathcal{V}ir_S^D$, we introduce the following notation:

- For $\mu, \nu \in \mathcal{T}_S$, the symbol $[\mu, \nu]$ denotes the natural Lie bracket in $\mathcal{T}_S$;
- For $\theta = (\theta_i)_{i=1}^L, \eta = (\eta_i)_{i=1}^L \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$, the symbol $[\theta, \eta]^0 = ([\theta_i, \eta_i]^0)_{i=1}^L$ denotes the natural Lie bracket in $\bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$ given by
$$[\theta_i(\xi_i)\partial_{\xi_i}, \eta_i(\xi_i)\partial_{\xi_i}]^0 = \big(\theta_i(\xi_i)\eta_i'(\xi_i) - \eta_i(\xi_i)\theta_i'(\xi_i)\big)\partial_{\xi_i}.$$
- $\mathrm{c_V}(\theta, \eta) := \sum_{i=1}^L \mathrm{Res}_{\xi_i=0}\big(\theta_i'''(\xi_i)\eta_i(\xi_i)\, d\xi_i\big)$. (The symbol $\mathrm{c_V}$ stands for "Cocycle defining the Virasoro algebra".);
- For $\theta \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$ and $f \in \mathcal{O}_S$, the symbols $\mu(\theta)$ and $\mu(f)$ denote the natural actions of a vector field $\mu \in \mathcal{T}_S$ on $\bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$ and $\mathcal{O}_S$ respectively.

We define the Lie algebra structure on $\mathcal{V}ir_S^D$ by
$$\begin{aligned}&[(\mu;\theta;f\hat{c}),(\nu;\eta;g\hat{c})]\\ &:= ([\mu,\nu];\mu(\eta)-\nu(\theta)+[\theta,\eta]^0;(\mu(g)-\nu(f)+\mathrm{c_V}(\theta,\eta))\hat{c}),\end{aligned} \tag{4.2}$$
where $(\mu;\theta;f\hat{c}), (\nu;\eta;g\hat{c}) \in \mathcal{V}ir_S^D$. Note that $\mathcal{V}ir_S^D$ is not an $\mathcal{O}_S$-Lie algebra but a $\mathbb{C}_S$-Lie algebra. We call $\mathcal{V}ir_S^D$ the *sheaf of Virasoro algebras on S.*

Remark 4.1. Later representations of $\mathcal{V}ir_S^D$ shall be interpreted as representations of an extension $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ of $\mathcal{T}_S$ defined below and thus shall be given a $\mathcal{D}_S$-module structure.

Let $\mathcal{T}_{\mathfrak{X}}$ denote the tangent sheaf of the total space $\mathfrak{X}$ and $\mathcal{T}_{\mathfrak{X}/S}$ the relative tangent sheaf of the family $\pi : \mathfrak{X} \to S$ (i.e., the sheaf of vector fields along the fibers of π on $\mathfrak{X}$). Since $\pi : \mathfrak{X} \to S$ is smooth, we have the following short exact sequence:
$$0 \to \mathcal{T}_{\mathfrak{X}/S}(*D) \to \mathcal{T}_{\mathfrak{X}}(*D) \to (\pi^*\mathcal{T}_S)(*D) \to 0.$$
Note that $\pi^*\mathcal{T}_S = \mathcal{O}_{\mathfrak{X}} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{T}_S$ does *not* possess a natural Lie algebra structure, but $\pi^{-1}\mathcal{T}_S \subset \pi^*\mathcal{T}_S$ does. Defining $\mathcal{T}_{\mathfrak{X},\pi}(*D)$ to be the inverse image of $\pi^{-1}\mathcal{T}_S$ in $\mathcal{T}_{\mathfrak{X}}(*D)$, we obtain the following Lie algebra extension:
$$0 \to \mathcal{T}_{\mathfrak{X}/S}(*D) \to \mathcal{T}_{\mathfrak{X},\pi}(*D) \to \pi^{-1}\mathcal{T}_S \to 0.$$
The direct image of this sequence by π is also exact and gives the following Lie algebra extension:
$$0 \to \mathcal{T}_{\dot{\mathfrak{X}}}^D \to \mathcal{V}ir_{\dot{\mathfrak{X}}}^D \to \mathcal{T}_S \to 0, \tag{4.3}$$
where we put
$$\mathcal{V}ir_{\dot{\mathfrak{X}}}^D := \pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D), \quad \mathcal{T}_{\dot{\mathfrak{X}}}^D := \pi_*\mathcal{T}_{\mathfrak{X}/S}(*D).$$

Remark 4.2. The exact sequence (4.3) is essential in the constructions of connections on the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks. Generally, a connection is defined to be an action of the tangent sheaf satisfying the certain axioms. Using the exact sequence (4.3), we can obtain a connection if we have an actions of $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ whose restriction on $\mathcal{T}_{\dot{\mathfrak{X}}}^D$ is trivial. (cf. Lemma 4.10, Lemma 4.11, Lemma 4.12, Lemma 4.13, Lemma 4.14, and Theorem 5.1.)

Lemma 4.3. *For a local section $a_{\dot{\mathfrak{X}}}$ of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} = \pi_* \mathcal{T}_{\mathfrak{X},\pi}(*D)$, its pull-back $\tilde{a}_{\dot{\mathfrak{X}}}$ to $\widetilde{\mathfrak{X}}$ is of the form:*

$$\tilde{a}_{\dot{\mathfrak{X}}} = \mu_0(\tau; z)\partial_\tau + \sum_{i=1}^{L} \mu_i(\tau; z)\partial_{z_i} + \theta^t(\tau; z; t)\partial_t,$$

where $\mu_i = \mu_i(\tau; z) \in \mathcal{O}_S$ and $\theta^t(\tau; z; t)$ is a meromorphic function globally defined along the fibers of $\tilde{\pi}$ with the following properties:

1. *The poles of $\theta^t(\tau; z; t)$ are contained in $\pi^{-1}_{\widetilde{\mathfrak{X}}/\mathfrak{X}}(D)$;*
2. *The quasi-periodicity:*

$$\theta^t(\tau; z; t + m\tau + n) = \theta^t(\tau; z; t) + m\mu_0(\tau; z) \quad \text{for } (m, n) \in \mathbb{Z}^2. \tag{4.4}$$

Proof. Let (m, n) be in $\mathbb{Z}^2$ and $f_{m,n}$ denote the action of (m, n) on $\widetilde{\mathfrak{X}}$ given by $(\tau; z; t) \mapsto (\tau; z; t + m\tau + n)$. Then its derivative $df_{m,n}$ sends ∂_τ, ∂_{z_i}, and ∂_t to $\partial_\tau + m\partial_t$, ∂_{z_i}, and ∂_t respectively. Since $\tilde{a}_{\dot{\mathfrak{X}}}$ induces the vector field $a_{\dot{\mathfrak{X}}}$ in $\pi_* \mathcal{T}_{\mathfrak{X},\pi}(*D)$, we obtain a formula

$$\mu + \theta^t(t + m\tau + n)\partial_t = df_{m,n}(\tilde{a}_{\dot{\mathfrak{X}}}),$$

which is equivalent to

$$\theta^t(t + m\tau + n) = \theta^t(t) + m\mu_0,$$

which proves the lemma. □

The local section $a_{\dot{\mathfrak{X}}}$ is mapped to $\mu = \mu_0\partial_t + \sum_{i=1}^L \mu_i\partial_{z_i} \in \mathcal{T}_S$ by the projection along π in (4.3) and belongs to $\mathcal{T}^D_{\dot{\mathfrak{X}}} = \pi_* \mathcal{T}_{\mathfrak{X}/S}(*D)$ if and only if $\mu = 0$. Under the local coordinate $\xi_i = t - z_i$, the local section $a_{\dot{\mathfrak{X}}}$ is uniquely represented in the following form:

$$a_{\dot{\mathfrak{X}}} = \mu + \theta_i(\xi_i)\partial_{\xi_i} \in \mathcal{T}_S \oplus \mathcal{O}_S((\xi_i))\partial_{\xi_i},$$

where $\theta_i(\xi_i)$ is the Laurent expansion of $\theta^t(\tau; z; t)$ in $\xi_i = t - z_i$. Thus we obtain the embedding of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ into $\mathcal{T}_S \oplus \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i} \subset \mathcal{V}ir^D_S$ given by

$$\mathcal{V}ir^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{T}_S \oplus \bigoplus_{i=1}^{L} \mathcal{O}_S((\xi_i))\partial_{\xi_i} \subset \mathcal{V}ir^D_S, \quad a_{\dot{\mathfrak{X}}} \mapsto (\mu; \theta) = (\mu; (\theta_i(\xi_i)\partial_{\xi_i})_{i=1}^L). \tag{4.5}$$

We identify $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ with its image in $\mathcal{V}ir^D_S$. For instance, $(\mu; \theta; 0) \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ means that $\mu \in \mathcal{T}_S$, $\theta \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$, and $(\mu; \theta; 0)$ belongs to the image of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ in $\mathcal{V}ir^D_S$. We also identify the subsheaf $\mathcal{T}^D_{\dot{\mathfrak{X}}} \subset \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ with its image in $\mathcal{V}ir^D_S$.

Remark 4.4. These formulations are essentially an application of the Beilinson-Schechtman theory in [BS] to our situation. The theory contains a natural construction of the Kodaira-Spencer map of a family of compact Riemann surfaces and its generalization to Virasoro algebras. For a brief sketch, see Appendix B.

A natural question is whether or not the embeddings of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ into $\mathcal{V}ir^D_S$ is a Lie algebra homomorphism.

Lemma 4.5. *The embedding $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$ is a Lie algebra homomorphism.*

Proof. Let $a_{\dot{\mathfrak{X}}}$ and $b_{\dot{\mathfrak{X}}}$ be in $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} = \pi_* \mathcal{T}_{\mathfrak{X},\pi}(*D)$. Denote their images in $\mathcal{V}ir^D_S$ by $(\mu; \theta; 0)$ and $(\nu; \eta; 0)$ and their pull-backs to $\widetilde{\mathfrak{X}}$ by $\tilde{a}_{\dot{\mathfrak{X}}}$ and $\tilde{b}_{\dot{\mathfrak{X}}}$ respectively. It suffices for the proof to show that $\mathrm{c_V}(\theta, \eta) = 0$.

Under the coordinate $(\tau; z; t)$ of $\widetilde{\mathfrak{X}}$, the vector fields $\tilde{a}_{\dot{\mathfrak{X}}}$ and $\tilde{b}_{\dot{\mathfrak{X}}}$ are represented as

$$\tilde{a}_{\dot{\mathfrak{X}}} = \mu + \theta^t(t)\partial_t, \quad \tilde{b}_{\dot{\mathfrak{X}}} = \nu + \eta^t(t)\partial_t, \tag{4.6}$$

where we write $\mu, \nu \in \mathcal{T}_S$ in the following forms:

$$\mu = \mu_0 \partial_\tau + \sum_{i=1}^{L} \mu_i \partial_{z_i}, \quad \nu = \nu_0 \partial_\tau + \sum_{i=1}^{L} \nu_i \partial_{z_i}. \tag{4.7}$$

Here we omit the arguments $(\tau; z)$ for simplicity: $\mu_i = \mu_i(\tau; z)$, $\theta^t(t) = \theta^t(\tau; z; t)$, etc. Because of Lemma 4.3, we have

$$\theta^t(t + m\tau + n) = \theta^t(t) + m\mu_0, \quad \eta^t(t + m\tau + n) = \eta^t(t) + m\nu_0.$$

Hence we can define the relative meromorphic 1-form $\omega \in \pi_* \Omega^1_{\mathfrak{X}/S}(*D)$ by

$$\omega = \omega(t)\, dt := \frac{\partial^2 \theta^t(t)}{\partial t^2} \frac{\partial \eta^t(t)}{\partial t}\, dt. \tag{4.8}$$

Here, the well-definedness of ω as a 1-form in $\pi_* \Omega^1_{\mathfrak{X}/S}(*D)$ follows from the fact that the definition of $\omega(t)$ implies $\omega(t + m\tau + n) = \omega(t)$ for $m, n \in \mathbb{Z}$.

On the other hand, under the local coordinate $(\tau; z; \xi_i)$ of $\mathfrak{X}$ around Q_i given by $\xi_i = t - z_i$, the vector fields $a_{\dot{\mathfrak{X}}}$ and $b_{\dot{\mathfrak{X}}}$ are represented in the following forms:

$$a_{\dot{\mathfrak{X}}} = \mu + \theta_i(\xi_i)\partial_{\xi_i}, \quad b_{\dot{\mathfrak{X}}} = \nu + \eta_i(\xi_i)\partial_{\xi_i},$$

where μ and ν are the same as those in (4.6) and θ_i and η_i are given in terms of θ^t, η^t, μ_i and ν_i in (4.6) and (4.7) by

$$\theta_i(\xi_i) = \theta^t(z_i + \xi_i) - \mu_i, \quad \eta_i(\xi_i) = \eta^t(z_i + \xi_i) - \nu_i.$$

Thus by (4.8), we have

$$\omega = \frac{\partial^2 \theta_i(\xi_i)}{\partial \xi_i^2} \frac{\partial \eta_i(\xi_i)}{\partial \xi_i}\, d\xi_i \quad \text{around } Q_i.$$

Hence the residue theorem leads to

$$\mathrm{c_V}(\theta, \eta) = -\sum_{i=1}^{L} \operatorname*{Res}_{\xi_i = 0} \left(\frac{\partial^2 \theta_i(\tau; z; \xi_i)}{\partial \xi_i^2} \frac{\partial \eta_i(\tau; z; \xi_i)}{\partial \xi_i}\, d\xi_i \right) = -\sum_{i=1}^{L} \operatorname{Res}_{Q_i} \omega = 0.$$

This proves the lemma. □

Remark 4.6. The same question about the embedding $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$ can be answered under a more general formulation for higher genus compact Riemann surfaces with a projective structure. However, in higher genus case, the embedding $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$ is not always a Lie algebra homomorphism. The case of genus 1 is very special. See Appendix C for a short sketch of a formulation.

In order to define the action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\hat{\mathfrak{g}}^D_S$, let us introduce the following notation:

- $A = (A_i)_{i=1}^L, B = (B_i)_{i=1}^L \in \bigoplus_{i=1}^L \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))$;

- $[A,B]^0 = ([A_i,B_i]^0)_{i=1}^L$ denotes the natural Lie bracket in $\mathfrak{g}_S^D \cong \bigoplus_{i=1}^L \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))$ given by the base extension of the Lie algebra $\mathfrak{g}$;
- $(A; f\hat{k}), (B; g\hat{k}) \in \hat{\mathfrak{g}}_S^D = \mathfrak{g}_S^D \oplus \mathcal{O}_S\hat{k}$;
- For $\mu \in \mathcal{T}_S$, the symbol $\mu(A)$ denotes the natural actions of $\mathcal{T}_S$ on $\mathfrak{g}_S^D$;
- The natural action of $\theta = (\theta_i)_{i=1}^L \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$ on $A \in \mathfrak{g}_S^D$ is defined by
$$\theta_i(A_i \otimes f_i(\xi_i)) := A_i \otimes (\theta_i(\xi_i) f_i'(\xi_i)) \quad \theta(A) := (\theta_i(A_i \otimes f_i))_{i=1}^L,$$
where $\theta_i = \theta_i(\xi_i)\partial_{\xi_i}$ and $A = A_i \otimes f_i(\xi_i) \in \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))$.

Then the action of $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ on $\hat{\mathfrak{g}}_S^D$ is defined by
$$(\mu;\theta;0)\cdot(A;g\hat{k}) := [(\mu;\theta;0),(A;g\hat{k})] \quad \text{for } (\mu;\theta;0) \in \mathcal{V}ir_{\dot{\mathfrak{X}}}^D \text{ and } (A;g\hat{k}) \in \hat{\mathfrak{g}}_S^D, \tag{4.9}$$
where $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ is identified with its image in $\mathcal{V}ir_S^D$ by (4.5) and the bracket of the right-hand side is a Lie bracket in the semi-direct product Lie algebra $\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D$ defined by
$$[(\mu;\theta;f\hat{c}),(A;g\hat{k})] := (\mu(A)+\theta(A);\mu(g)\hat{k}) \quad \text{for } (\mu;\theta;f\hat{c}) \in \mathcal{V}ir_S^D \text{ and } (A;g\hat{k}) \in \hat{\mathfrak{g}}_S^D. \tag{4.10}$$

Lemma 4.7. *The action of $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ on $\hat{\mathfrak{g}}_S^D$ preserves $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$.*

Proof. Because of (3.16), under the identifications above, the restriction of the action of $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D = \pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D)$ on $\mathfrak{g}_{\dot{\mathfrak{X}}}^D$ comes from the action of $\mathcal{T}_{\mathfrak{X},\pi}(*D)$ on $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*D)$ via the connection ∇ on $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}$. Namely, if $a_{\dot{\mathfrak{X}}} \in \mathcal{V}ir_{\dot{\mathfrak{X}}}^D$, $A_{\dot{\mathfrak{X}}} \in \mathfrak{g}_{\dot{\mathfrak{X}}}^D$, and their images in $\mathcal{V}ir_S^D$ and $\hat{\mathfrak{g}}_S^D$ are denoted by $(\mu;\theta;0)$ and $(A;0)$ respectively, then
$$[(\mu;\theta;0),(A;0)] = \left(\text{the image of } \nabla_{a_{\dot{\mathfrak{X}}}} A_{\dot{\mathfrak{X}}} \in \pi_*\mathfrak{g}^{\mathrm{tw}}(*D)\right).$$
Thus we obtain $[\mathcal{V}ir_{\dot{\mathfrak{X}}}^D, \mathfrak{g}_{\dot{\mathfrak{X}}}^D] \subset \mathfrak{g}_{\dot{\mathfrak{X}}}^D$. ∎

4.2. **Action of the sheaf of Virasoro algebras.** In this subsection we define an action of the Lie algebra $\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D$ on $\hat{\mathfrak{g}}_S^D$ modules.

Fix an arbitrary complex number k. For each $i = 1, \dots, L$, let M_i be a representation with level k of the affine Lie algebra $\hat{\mathfrak{g}}_i$ satisfying the smoothness condition (3.24). Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Then M is a representation with level k of the affine Lie algebra $(\mathfrak{g}^{\oplus L})^\wedge = \bigoplus_{i=1}^L \left(\mathfrak{g} \otimes \mathbb{C}((\xi_i))\right) \oplus \mathbb{C}\hat{k}$ and $\mathcal{M}$ is a $\hat{\mathfrak{g}}_S^D$-module.

The Sugawara operators $S[m]$ acting on M_i's are given by the formula (1.47) and its action on the i-th factor M_i in M is denoted by $\rho_i(S[m])$. Define the Sugawara tensor field by
$$S(\xi)(d\xi)^2 := \sum_{m\in\mathbb{Z}} \xi^{-m-2} S[m]\,(d\xi)^2,$$
and its action on M_i is denoted by $\rho_i(S(\xi_i))(d\xi)^2$. Then, by the same way as Lemma 1.11, we can prove the following lemma.

Lemma 4.8. *For any $s \in S$, $\Phi \in \mathcal{CB}(\mathcal{M})_s$ and $v \in \mathcal{M}_s$, there exists a unique $\omega \in (\pi_*\Omega^2_{\mathfrak{X}/S}(*D))_s$ such that the expression of ω under the coordinate ξ_i coincides with $\Phi(\rho_i(S(\xi_i))v)\,(d\xi_i)^2$ for each $i = 1, \dots, L$.*

We denote ω in Lemma 4.8 by $\Phi(S(\xi)v)\,(d\xi)^2$ or $\Phi(S(P)v)\,(dP)^2$, which is called a *correlation function of the Sugawara tensor* $S(\xi)$ and v under Φ, or a *Sugawara correlation function* for short.

Assume that $\kappa = k + h^\vee \neq 0$ and put $c_k := k \dim \mathfrak{g}/\kappa$. Define the Virasoro operators $T[m]$ and the energy-momentum tensor $T(\xi)$ by

$$T[m] := \kappa^{-1} S[m], \quad T(\xi)(d\xi)^2 := \kappa^{-1} S(\xi)(d\xi)^2 \tag{4.11}$$

as in (1.50) and the *energy-momentum correlation function* $\Phi(T(\xi)v)\,(d\xi)^2$ to be $\kappa^{-1}\Phi(S(\xi)v)\,(d\xi)^2$ as in (1.41).

The action $\rho_i(T[m])$ of the Virasoro operators on M_i defines a representation of the Virasoro algebra with central charge $c_k = k \dim \mathfrak{g}/\kappa$ (Lemma 1.13). For $v_i \otimes g \in M_i \otimes \mathcal{O}_S$ and $\theta_i = \sum_{m\in\mathbb{Z}} \theta_{i,m} \xi_i^{m+1} \partial_{\xi_i} \in \mathcal{O}_S((\xi_i))\partial_{\xi_i}$, put

$$\rho_i(T\{\theta_i\})(v_i \otimes g) = \sum_{m\in\mathbb{Z}} (\rho_i(-T[m])v_i) \otimes (\theta_{i,m} g). \tag{4.12}$$

For example, $\rho_i(T\{\xi_i^{m+1}\partial_{\xi_i}\}) = \rho_i(-T[m])$. For $\theta = (\theta_i)_{i=1}^L \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$, the operator $T\{\theta\}$ acting on $\mathcal{M}$ is defined by

$$T\{\theta\} := \sum_{i=1}^L \rho_i(T\{\theta_i\}), \tag{4.13}$$

where we consider $\rho_i(T\{\theta_i\})$ as an operator acting on the i-th factor in $\mathcal{M}$.

Define the action of $(\mu; \theta; f\hat{c}) \in \mathcal{V}ir_S^D$ on $\mathcal{M}$ by

$$(\mu; \theta; f\hat{c}) \cdot (v \otimes g) := v \otimes \mu(g) + T\{\theta\}(v \otimes g) + c_k v \otimes (fg) \tag{4.14}$$

for $v \otimes g \in \mathcal{M} = M \otimes \mathcal{O}_S$. The dual actions on $\mathcal{M}^* := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}, \mathcal{O}_S)$ are defined by

$$(\mu\Phi)(v) := \mu(\Phi(v)) - \Phi(\mu(v)), \tag{4.15}$$
$$(\rho_i^*(T^*\{\theta_i\}\Phi))(v) := -\Phi(\rho_i(T\{\theta_i\})v), \tag{4.16}$$
$$T^*\{\theta\} := \sum_{i=1}^L \rho_i^*(T\{\theta_i\}), \tag{4.17}$$
$$((\mu; \theta; f\hat{c}) \cdot \Phi)(v) := \mu(\Phi(v)) - \Phi((\mu; \theta; f\hat{c}) \cdot v), \tag{4.18}$$

where $\Phi \in \mathcal{M}^*$, and $v \in \mathcal{M}$. Since we have

$$(\mu; \theta; f\hat{c}) \cdot \Phi = \mu\Phi + T^*\{\theta\}\Phi - c_k f \Phi, \tag{4.19}$$

the action of $\mathcal{V}ir_S^D$ on $\mathcal{M}^*$ defines a representation of $\mathcal{V}ir_S^D$ with central charge $-c_k$.

The Virasoro operators $T[m]$ satisfy the commutation relations (1.52). Therefore a straightforward calculation proves the following lemma.

Lemma 4.9. *The action of* $\hat{\mathfrak{g}}_S^D$ (3.25) *and that of* $\mathcal{V}ir_S^D$ (4.14) *on* $\mathcal{M}$ *induce a representation of the Lie algebra* $\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D$*, whose semi-direct product Lie algebra structure is given by* (4.10)*.*

Define the actions of $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ on $\mathcal{M}$ and $\mathcal{M}^*$ through the embedding $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D \hookrightarrow \mathcal{V}ir_S^D$ and the actions of $\mathcal{V}ir_S^D$. Then Lemma 4.5 and Lemma 4.9 immediately lead to the following lemma.

Lemma 4.10. *These actions of* $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ *on* $\mathcal{M}$ *and* $\mathcal{M}^*$ *are representations of the Lie algebra* $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$*.*

Lemma 4.11. *The action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}$ preserves $\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$ and hence defines a representation on $\mathcal{CC}(\mathcal{M})$ of the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$.*

Proof. Assume that $\alpha_{\dot{\mathfrak{X}}} \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$, $A_{\dot{\mathfrak{X}}} \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$, and $v \in \mathcal{M}$. Lemma 4.9 implies that

$$\alpha_{\dot{\mathfrak{X}}} A_{\dot{\mathfrak{X}}} v = [\alpha_{\dot{\mathfrak{X}}}, A_{\dot{\mathfrak{X}}}]v + A_{\dot{\mathfrak{X}}}\alpha_{\dot{\mathfrak{X}}} v,$$

and Lemma 4.7 means that $[\alpha_{\dot{\mathfrak{X}}}, A_{\dot{\mathfrak{X}}}] \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$. Hence $\alpha_{\dot{\mathfrak{X}}} A_{\dot{\mathfrak{X}}} v$ belongs to $\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$ and $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$ is included in $\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$. From Lemma 4.10 it follows that the induced action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{CC}(\mathcal{M}) = \mathcal{M}/\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$ defines a representation of the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$. □

As a result of Lemma 4.10, Lemma 4.11 and (4.18), we obtain the following lemma.

Lemma 4.12. *The action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}^*$ preserves the subsheaf $\mathcal{CB}(\mathcal{M})$ of $\mathcal{M}^*$ and defines a representation on $\mathcal{CB}(\mathcal{M})$ of the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$.*

The actions of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}$ and $\mathcal{M}^*$ are also defined through the embedding $\mathcal{T}^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$. Then, as in the proof of Lemma 1.14, we can show the following lemma from Lemma 4.8 thanks to the existence of the energy-momentum correlation function (4.11).

Lemma 4.13. *The action of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}^*$ satisfies $\mathcal{T}^D_{\dot{\mathfrak{X}}} \cdot \mathcal{CB}(\mathcal{M}) = 0$.*

Using the exact sequence (4.3), Lemma 4.12, and Lemma 4.13, we can construct a flat connection on the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks in §5 (Remark 5.2). However, for the construction of a flat connection on the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants, we shall need the following lemma, as well as the exact sequence (4.3) and Lemma 4.11.

Lemma 4.14. *The action of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}$ satisfies $\mathcal{T}^D_{\dot{\mathfrak{X}}}\mathcal{M} \subset \mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$.*

We remark that Lemma 4.14 implies Lemma 4.13, but the converse does not hold. The key point in the proof of Lemma 4.13 is the notion of the energy-momentum correlation function, which is not useful for the proof of Lemma 4.14. Hence we must find a direct proof of Lemma 4.14 without using the energy-momentum correlation functions. The rest of this subsection is devoted to the proof of this lemma along the course similar to that of [Ts].

We define the $\mathcal{O}_S$-inner product $(\,.\,,.\,)$ on $\mathfrak{g}^D_S \cong \bigoplus_{i=1}^L \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))$ by

$$\big((A_i \otimes f_i)_{i=1}^L, (B_i \otimes g_i)_{i=1}^L\big) = \sum_{i=1}^L (A_i|B_i) \operatorname*{Res}_{\xi_i=0}(f_i g_i \, d\xi_i) \tag{4.20}$$

for $A_i, B_i \in \mathfrak{g}$ and $f_i, g_i \in \mathcal{O}_S((\xi_i))$. This inner product is non-degenerate and allows us to regard $\mathfrak{g}^D_S$ as the topological dual of itself under the ξ_i-adic topologies. Putting

$$R := \{\, (a,b,m,i) \mid (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0,0)\},\ m \in \mathbb{Z},\ i = 1,\dots,L \,\}, \tag{4.21}$$

$$e^m_{a,b,i} = (e^m_{a,b,i,j}(\xi_j))_{j=1}^L := (\delta_{i,j} J_{a,b} \otimes \xi_j^m)_{j=1}^L, \tag{4.22}$$

$$e_m^{a,b,i} = (e^{a,b,i}_{m,j}(\xi_j))_{j=1}^L := (\delta_{i,j} J^{a,b} \otimes \xi_j^{-m-1})_{j=1}^L, \tag{4.23}$$

we obtain the following topological dual $\mathcal{O}_S$-bases of $\mathfrak{g}_S^D$ with respect to the inner product:

$$F_0 := \{\, e^m_{a,b,i} \mid (a,b,m,i) \in R \,\}, \quad F^0 := \{\, e_m^{a,b,i} \mid (a,b,m,i) \in R \,\}.$$

For $A = (A_i)_{i=1}^L, B = (B_i)_{i=1}^L \in \mathfrak{g}_S^D$, we introduce the following notation:

$$\rho(A) := \sum_{i=1}^L \rho_i(A_i), \tag{4.24}$$

$$\mathop{\circ}\limits_{\circ}\rho(A)\rho(B)\mathop{\circ}\limits_{\circ} := \sum_{i=1}^L \rho_i(\mathop{\circ}\limits_{\circ} A_i B_i \mathop{\circ}\limits_{\circ}) + \sum_{i\neq j} \rho_i(A_i)\rho_j(B_j). \tag{4.25}$$

Recall that, for $\theta = (\theta_i(\xi_i)\partial_{\xi_i})_{i=1}^L \in \mathcal{T}_S^D$, the Virasoro operator $T\{\theta\}$ acting on $\mathcal{M}$ is defined by (4.13), (4.12), (4.11) and (1.47). Using the dual bases above, we can represent the Virasoro operator $T\{\theta\}$ in the following form:

$$T\{\theta\} = -\frac{1}{2\kappa} \sum_{(a,b,m,i)\in R} \mathop{\circ}\limits_{\circ}\rho(e_m^{a,b,i}\circ\theta)\rho(e^m_{a,b,i})\mathop{\circ}\limits_{\circ}, \tag{4.26}$$

where we put

$$e_m^{a,b,i} \circ \theta := (e_{m,j}^{a,b,i}(\xi_j)\theta_j(\xi_j))_{j=1}^L \in \mathfrak{g}_S^D.$$

The formula (4.26) follows from the special cases with $\theta = (\delta_{i,j}\xi_j^{n+1}\partial_{\xi_j})_{j=1}^L$ for $n \in \mathbb{Z}$, which are obtained by straightforward calculations.

The bases which we really need later are however not these naively defined bases, F_0 and F^0, but "good" dual frames in the sense of [Ts]. See Lemma 4.15 and Remark 4.16 below. In order to construct such dual bases, we define the meromorphic functions $w^n_{a,b}$ on $\mathfrak{H} \times \mathbb{C}$ for $n = 0, 1, 2, \dots$ by derivatives of $w_{a,b}$ (1.14):

$$w^n_{a,b}(t) = w^n_{a,b}(\tau; t) := \frac{(-1)^n}{n!}\frac{\partial^n}{\partial t^n} w_{a,b}(\tau; t). \tag{4.27}$$

Then the meromorphic function $w^n_{a,b}(\tau; t - z_i)$ on $\widetilde{\mathfrak{X}}$ can be regarded as a global section of the line bundle $L_{a,b}(*Q_i)$ on $\mathfrak{X}$ and its Laurent expansion in ξ_i is written in the following form:

$$w^n_{a,b}(\xi_i) = \xi_i^{-n-1} + (-1)^n w_{a,b,n} + O(\xi_i), \tag{4.28}$$

where $w_{a,b,n}$ is the coefficient of t^n in the Laurent expansion (1.15) of $w_{a,b}(t)$. For $(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 \smallsetminus \{(0,0)\}$, $i, j = 1, \dots, L$, and $m \in \mathbb{Z}$, we define $f^m_{a,b,i,j} \in \mathcal{O}_S((\xi_j))$ by

$$f^m_{a,b,i,j}(\xi_j) := \begin{cases} \delta_{i,j}\xi_j^m & \text{if } m \geqq 0, \\ w_{a,b}^{-m-1}(z_j - z_i + \xi_j) & \text{if } m < 0, \end{cases} \tag{4.29}$$

and put

$$\begin{aligned} J^m_{a,b,i} &= (J^m_{a,b,i,j})_{j=1}^L := (J_{a,b} \otimes f^m_{a,b,i,j}(\xi_j))_{j=1}^L, \\ J_m^{a,b,i} &= (J_{m,j}^{a,b,i})_{j=1}^m := (J^{a,b} \otimes f^{-m-1}_{-a,-b,i,j}(\xi_j))_{j=1}^L. \end{aligned} \tag{4.30}$$

Then the following topological $\mathcal{O}_S$-bases of $\mathfrak{g}_S^D$

$$F_1 := \{\, J^m_{a,b,i} \mid (a,b,m,i) \in R \,\}, \quad F^1 := \{\, J_m^{a,b,i} \mid (a,b,m,i) \in R \,\}, \tag{4.31}$$

are dual to each other by virtue of the residue theorem. Changing the bases of expansion from F_0 and F^0 to F_1 and F^1, we obtain the another expression of $T\{\theta\}$ from (4.26):

$$T\{\theta\} = -\frac{1}{2\kappa} \sum_{(a,b,m,i)\in R} {}^{\circ}_{\circ} \rho(J^{a,b,i}_m \circ \theta)\rho(J^m_{a,b,i}) {}^{\circ}_{\circ}, \tag{4.32}$$

where we put, as above,

$$J^{a,b,i}_m \circ \theta := (J^{a,b,i}_{m,j}(\xi_j)\theta_j(\xi_j))^L_{j=1} \in \mathfrak{g}^D_S.$$

The following lemma immediately follows from the definitions above and the decomposition (3.9) of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ to the direct sum of the line bundles $L_{a,b}$.

Lemma 4.15. *Under the notation above, we have the following:*

1. *If m is negative, then $J^m_{a,b,i}$ is equal to the image of $J_{a,b}w^m_{a,b}(\tau; t - z_i) \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$.*
2. *If m is not negative and θ is a local section of $\mathcal{T}^D_{\dot{\mathfrak{X}}} = \pi_* \mathcal{T}_{\mathfrak{X}/S}(*D)$ with $\theta = \theta^t(\tau; z; t)\partial_t$, then $J^{a,b,i}_m \circ \theta$ is equal to the image of $J^{a,b}w^{-m-1}_{-a,-b}(\tau; t - z_i)\theta^t(\tau; z; t) \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}$.*

Remark 4.16. This lemma means that the topological dual bases F_1 and F^1 given by (4.31) are *good dual frames* of $\mathfrak{g}^D_S$ in the sense of Tsuchimoto [Ts].

For $f_i \in \mathcal{O}_S((\xi_i))$, the regular part $f_{i,+}$ and the singular part $f_{i,-}$ are uniquely defined by the conditions

$$f_i = f_{i,+} + f_{i,-}, \quad f_{i,+} \in \mathcal{O}_S[[\xi_i]], \quad f_{i,-} \in \xi_i^{-1}\mathcal{O}_S[\xi_i^{-1}].$$

Note that the differentiation ∂_{ξ_i} commutes the operations $f_i \mapsto f_{i,\pm}$:

$$\partial_{\xi_i}(f_{i,\pm}(\xi_i)) = (f'_i(\xi_i))_\pm,$$

which shall be denoted by $f'_{i,\pm}(\xi_i)$.

Lemma 4.17. *For $f_i, g_i \in \mathcal{O}_S((\xi_i))$, the normal product ${}^{\circ}_{\circ}(J^{a,b} \otimes f_i)(J_{a,b} \otimes g_i){}^{\circ}_{\circ}$ can be represented in the following forms:*

$$\begin{aligned}
{}^{\circ}_{\circ}(J^{a,b} \otimes f_i)(J_{a,b} \otimes g_i){}^{\circ}_{\circ} &= (J^{a,b} \otimes f_i)(J_{a,b} \otimes g_i) + \hat{k} \operatorname*{Res}_{\xi_i=0}(f_{i,+}(\xi_i)g'_{i,-}(\xi_i)\,d\xi_i) \\
&= (J_{a,b} \otimes g_i)(J^{a,b} \otimes f_i) - \hat{k} \operatorname*{Res}_{\xi_i=0}(f_{i,-}(\xi_i)g'_{i,+}(\xi_i)\,d\xi_i).
\end{aligned}$$

Proof. From the commutativity of $J_{a,b}$ and $J^{a,b}$ and the definition of the normal product (1.45), we can find that

$${}^{\circ}_{\circ}(J^{a,b} \otimes f_i)(J_{a,b} \otimes g_i){}^{\circ}_{\circ} = (J^{a,b} \otimes f_i)(J_{a,b} \otimes g_{i,+}) + (J_{a,b} \otimes g_{i,-})(J^{a,b} \otimes f_i).$$

Using the definition (3.19) of the Lie algebra structure on $\hat{\mathfrak{g}}^D_S$, we obtain the formulae

$$[J^{a,b} \otimes g_{i,-}, J_{a,b} \otimes f_i] = \hat{k} \operatorname*{Res}_{\xi_i=0}\big(g'_{i,-}(\xi_i)f_i(\xi_i)\,d\xi_i\big) = \hat{k} \operatorname*{Res}_{\xi_i=0}\big(f_{i,+}(\xi_i)g'_{i,-}(\xi_i)\,d\xi_i\big), \tag{4.33}$$

$$[J^{a,b} \otimes f_i, J_{a,b} \otimes g_{i,+}] = \hat{k} \operatorname*{Res}_{\xi_i=0}\big(f'_i(\xi_i)g_{i,+}(\xi_i)\,d\xi_i\big) = -\hat{k} \operatorname*{Res}_{\xi_i=0}\big(f_{i,-}(\xi_i)g'_{i,+}(\xi_i)\,d\xi_i\big). \tag{4.34}$$

These formulae prove the lemma. ∎

For the brevity of notation, we introduce the sets $R_\pm$ by

$$R_+ := \{\, (a,b,m,i) \in R \mid m \geqq 0 \,\}, \quad R_- := \{\, (a,b,m,i) \in R \mid m < 0 \,\},$$

where R is defined by (4.21).

Lemma 4.18. *For $\theta = (\theta_i(\xi_i)\partial_{\xi_i})_{i=1}^L \in \bigoplus_{i=1}^L \mathcal{O}_S((\xi_i))\partial_{\xi_i}$, we have the following expressions of the Virasoro operator $T\{\theta\}$:*

$$T\{\theta\} = -\frac{1}{2\kappa}\Big(\sum_{R_+} \rho(J^{a,b,i}_m \circ \theta)\rho(J^m_{a,b,i}) + \sum_{R_-} \rho(J^m_{a,b,i})\rho(J^{a,b,i}_m \circ \theta)\Big), \tag{4.35}$$

where the symbols $\sum\limits_{R_\pm}$ denote the summations over $(a,b,m,i) \in R_\pm$. Moreover, substituting the definition (4.30) to this formula, we obtain

$$\begin{aligned} T\{\theta\} = -\frac{1}{2\kappa}\Big(&\sum_{R_+}\sum_{j=1}^L \rho_j(J^{a,b} \otimes w^m_{-a,-b}(z_j - z_i + \xi_j)\theta_j(\xi_j))\, \rho_i(J_{a,b} \otimes \xi_i^m) \\ &+ \sum_{R_-} \rho(J^m_{a,b,i})\rho(J^{a,b,i}_m \circ \theta)\Big). \end{aligned} \tag{4.36}$$

Proof. From the definitions (4.27), (4.29), and (4.30), we can derive the following formulae:

$$\begin{aligned} &\rho(J^m_{a,b,i}) = \rho_i(J_{a,b} \otimes \xi_i^m) && \text{for } (a,b,m,i) \in R_+, \\ &\rho(J^{a,b,i}_m \circ \theta) = \rho_i(J^{a,b} \otimes \xi_i^{-m-1}\theta_i(\xi_i)) && \text{for } (a,b,m,i) \in R_-, \\ &J^m_{a,b,i,i} = J_{a,b} \otimes w^{-m-1}_{a,b}(\xi_i) && \text{for } (a,b,m,i) \in R_-, \\ &\partial_{\xi_i}(w^{-m-1}_{a,b,+}(\xi)) = m w^{-m}_{a,b,+}(\xi_i) && \text{for } (a,b,m,i) \in R_-. \end{aligned}$$

Therefore, applying Lemma 4.17 to the formula (4.32), we obtain

$$T\{\theta\} = (\text{the right-hand side of (4.35)}) + \hat{k}R(\theta),$$

where $R(\theta)$ is defined by

$$R(\theta) := \sum_{R_-} \operatorname*{Res}_{\xi_i=0}\big((\xi_i^{-m-1}\theta_i(\xi_i))_-\,(-m)w^{-m}_{a,b,+}(\xi_i)\, d\xi_i\big),$$

which is a finite sum and hence is a local section of $\mathcal{O}_S$. Hence, for the proof of (4.35), it is enough to show that $R(\theta) = 0$ in the case of $\theta = (\delta_{i,j}\xi_j^{-l}\partial_{\xi_j})_{j=1}^L$ for $l = 1,2,3,\dots$ and $i = 1,\dots,L$. Using the formula (4.27) and the Laurent expansion (1.15) of $w_{a,b}(t)$, we can find that

$$\begin{aligned} &\sum_{m<0} \operatorname*{Res}_{\xi_i=0}\big((\xi_i^{-m-1}\xi_i^{-l})_-\,(-m)w^{-m}_{a,b,+}(\xi_i)\, d\xi_i\big) \\ &= \sum_{n=1}^{l} \operatorname*{Res}_{\xi_i=0}\left(\xi_i^{n-l-1}\, n(-1)^n \sum_{\nu \geqq n} \binom{\nu}{n} w_{a,b,\nu}\xi_i^{\nu-n}\, d\xi_i\right) \\ &= \left(\sum_{n=1}^{l}(-1)^n n\binom{l}{n}\right) w_{a,b,l} = \begin{cases} -w_{a,b,1} & \text{if } l = 1, \\ 0 & \text{if } l > 2. \end{cases} \end{aligned}$$

Hence we obtain $R(\theta) = 0$ by (1.18). We have proved the first expression (4.35) of $T\{\theta\}$. $\square$

Remark 4.19. The second expression (4.36) of $T\{\theta\}$ is very useful for the explicit calculations of the elliptic Knizhnik-Zamolodchikov connections (§5.2).

We are ready to prove Lemma 4.14. Let v be in $\mathcal{M}$ and θ in $\mathcal{T}^D_{\dot{\mathfrak{X}}}$. Then, applying Lemma 4.15 to the first expression (4.35) of $T\{\theta\}$ in Lemma 4.18 shows that $T\{\theta\}v \in \mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$. Hence we have proved Lemma 4.14.

5. Flat connections

We keep all the notation in the previous section. In this section, we assume $\kappa = k + h^\vee \neq 0$ and shall construct $\mathcal{D}_S$-module structures on the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and on the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks. We can show as a direct consequence that, under the assumption of Lemma 3.5, the sheaves $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ are locally free coherent $\mathcal{O}_S$-modules (i.e., vector bundles on S) with flat connections and their fibers at $s \in S$ are canonically isomorphic to the space of conformal coinvariants and that of conformal blocks respectively. In §5.2, we shall show that the connections on $\mathcal{CC}_k(V)$ and $\mathcal{CB}_k(V)$ coincide with the elliptic Knizhnik-Zamolodchikov equations introduced by Etingof [E]. In §5.3, we shall obtain a proof of the modular property of the connections without referring the explicit expressions of them.

5.1. **Construction of flat connections.** Recall that we have the Lie algebra extension $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ of the tangent sheaf $\mathcal{T}_S$ by $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ in (4.3). Since the action of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ maps $\mathcal{M}$ into $\mathfrak{g}^D_{\dot{\mathfrak{X}}}\mathcal{M}$ due to Lemma 4.14, the representation of the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{CC}(\mathcal{M})$ given by Lemma 4.11 induces the Lie algebra action of $\mathcal{T}_S$ on $\mathcal{CC}(\mathcal{M})$, which shall be denoted by

$$\mathcal{T}_S \times \mathcal{CC}(\mathcal{M}) \to \mathcal{CC}(\mathcal{M}), \quad (\mu, \phi) \mapsto D_\mu \phi. \tag{5.1}$$

Moreover it immediately follows from (4.14) that

$$D_{f\mu} v = f(D_\mu v), \quad D_\mu(fv) = \mu(f)v + f(D_\mu v)$$

for $f \in \mathcal{O}_S$, $\mu \in \mathcal{T}_S$, and $v \in \mathcal{CC}(\mathcal{M})$. Thus we obtain the flat connection D on the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants. Because of $\mathcal{CB}(\mathcal{M}) = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{CC}(\mathcal{M}), \mathcal{O}_S)$, we obtain the dual connection D^* on $\mathcal{CB}(\mathcal{M})$:

$$(D^*_\mu \Phi)(v) := \mu(\Phi(v)) - \Phi(D_\mu v). \tag{5.2}$$

We can summarize the results as follows.

Theorem 5.1. *For each $i = 1, \dots, L$, let M_i be a representation with level k of the affine Lie algebra $\hat{\mathfrak{g}}_i$ satisfying the smoothness condition* (3.24). *Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Then M is a representation with level k of $(\mathfrak{g}^{\oplus L})^\wedge$ and $\mathcal{M}$ is a $\hat{\mathfrak{g}}^D_S$-module. Assume that $\kappa = k + h^\vee \neq 0$. Then the action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}$ induces the $\mathcal{D}_S$-module structures on the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and on the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks.*

Remark 5.2. We have another description of the dual connection D^* on $\mathcal{CB}(\mathcal{M})$. Lemma 4.13 shows that the representation of the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{CB}(\mathcal{M})$ given by Lemma 4.12 induces the Lie algebra action of $\mathcal{T}_S$ on $\mathcal{CB}(\mathcal{M})$, which coincides with the dual connection D^*.

Corollary 5.3. *For each $i = 1, \dots, L$, let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ and M_i a quotient module of the Weyl module $M_k(V_i)$ over the affine Lie algebra $\hat{\mathfrak{g}}_i$. Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Assume that $\kappa = k + h^\vee \neq 0$. Then the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks are locally free coherent $\mathcal{O}_S$-modules with flat connections on S and dual to each other.*

Proof. We have already shown the $\mathcal{O}_S$-coherencies of $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ in Corollary 3.5 and the existences of $\mathcal{D}_S$-module structures on $\mathcal{CC}(\mathcal{M})$ and on $\mathcal{CB}(\mathcal{M})$ in Theorem 5.1. It is well-known that any $\mathcal{O}_S$-coherent $\mathcal{D}_S$-module is $\mathcal{O}_S$-locally free. (See Theorem 6.1 in Chapter I of [Ho], Proposition 1.7 in Chapter VI of [BEGHKM], or Theorem 1.1.25 of [Bj].) □

Corollary 5.4. *For each $i = 1, \dots, L$, let M_i be a representation with level k of the affine Lie algebra $\hat{\mathfrak{g}}_i$ satisfying the smoothness condition* (3.24). *Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Then the fiber of $\mathcal{CC}(\mathcal{M})$ at $s = (\tau; z) \in S$ is canonically isomorphic to the space of conformal coinvariants for $(X; \{Q_i\}_{i=1}^L) = (X_\tau; \{q_i(s)\}_{i=1}^L)$:*

$$\mathcal{CC}(\mathcal{M})|_s \cong \mathrm{CC}(M). \tag{5.3}$$

Moreover, under the assumptions of Corollary 5.3, the fiber of $\mathcal{CB}(\mathcal{M})$ at $s \in S$ is canonically isomorphic to the space of conformal blocks:

$$\mathcal{CB}(\mathcal{M})|_s \cong \mathrm{CB}(M). \tag{5.4}$$

Proof. We can identify the restriction of $\mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}$ on $X = \pi^{-1}(s)$ with $\mathfrak{g}^{\mathrm{tw}}$ in §1. Put $Q := D \cap X$. Then we can find the following canonical isomorphisms without using Corollary 5.3:

$$\mathcal{M}|_s = (M \otimes \mathcal{O}_S)|_s \cong M \otimes \mathbb{C} = M, \tag{5.5}$$

$$\left.(\pi_* \mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(mD))\right|_s \cong H^0(X, \mathfrak{g}^{\mathrm{tw}}(mQ)) \quad \text{for } m \geqq 0, \tag{5.6}$$

$$\left.\mathfrak{g}_{\dot{\mathfrak{X}}}^D\right|_s = \left.(\pi_* \mathfrak{g}_{\mathfrak{X}}^{\mathrm{tw}}(*D))\right|_s \cong H^0(X, \mathfrak{g}^{\mathrm{tw}}(*Q)) = \mathfrak{g}_{\dot{X}}^Q, \tag{5.7}$$

$$\left.(\mathfrak{g}_{\dot{\mathfrak{X}}}^D \mathcal{M})\right|_s \cong \mathfrak{g}_{\dot{X}}^Q M. \tag{5.8}$$

The Riemann-Roch theorem shows that $\dim_{\mathbb{C}} H^0(\pi^{-1}(s), \mathfrak{g}^{\mathrm{tw}}(mD)|_{\pi^{-1}(s)})$ is a constant function of $s \in S$ if $m \geqq 0$. Therefore the existence of the isomorphism (5.6) follows from the Grauert theorem. (For the Grauert theorem, see, for example, Corollary 12.9 in Chapter III of [Har] and Theorem in Chapter 10 §5.3 (p. 209) of [GR]). The isomorphism (5.7) is obtained by the inductive limit of (5.6). The isomorphism (5.8) is obtained by using (5.5), (5.7), and applying the right exact functor $(\cdot)|_s$ to the exact sequence $\mathfrak{g}_{\dot{\mathfrak{X}}}^D \otimes_{\mathcal{O}_S} \mathcal{M} \to \mathfrak{g}_{\dot{\mathfrak{X}}}^D \mathcal{M} \to 0$. Similarly the isomorphism (5.3) is obtained from (5.5) and (5.8).

Under the assumption of Corollary 5.3, the sheaves $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ are locally free $\mathcal{O}_S$-modules of finite rank and dual to each other. Then we have $\mathcal{CB}(\mathcal{M})|_s = \mathrm{Hom}_{\mathbb{C}}(\mathcal{CC}(\mathcal{M})|_s, \mathbb{C})$. This formula together with the isomorphism (5.3) gives the isomorphism (5.4). □

Let us describe the connections D and D^* more explicitly. For any vector field $\mu \in \mathcal{T}_S$, the action of D_μ on $\mathcal{CC}(\mathcal{M})$ and that of D_μ^* on $\mathcal{CB}(\mathcal{M})$ are described as follows. Using the short exact sequence (4.3), we can lift μ to an element $(\mu; \theta; 0) \in$

$\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ at least locally on S, and the ambiguity in the choice of $(\mu;\theta;0)$ is equal to $\mathcal{T}^D_{\dot{\mathfrak{X}}}$. But, since the actions of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ on $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ are trivial, the actions of $(\mu;\theta;0)$ on $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ do not depend on the choice of the lift and give D_μ and D^*_μ:

$$D_\mu v = (\mu;\theta;0)\cdot v = \mu(v) + T\{\theta\}v, \tag{5.9}$$

$$D^*_\mu \Phi = (\mu;\theta;0)\cdot \Phi = \mu(\Phi) + T^*\{\theta\}\Phi \tag{5.10}$$

for $(\mu;\theta;0) \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ and $v \in \mathcal{CC}(\mathcal{M})$, and $\Phi \in \mathcal{CB}(\mathcal{M})$. The following lemma provides us with the explicit formulae of $(\mu;\theta;0) \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$.

Lemma 5.5. *The following expressions in the coordinate $(\tau;z;t)$ define vector fields in $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} = \pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D)$ and are lifts of ∂_{z_i} and ∂_τ respectively:*

$$a_{\dot{\mathfrak{X}}}(\partial_{z_i}) = \partial_{z_i}, \quad a_{\dot{\mathfrak{X}}}(\partial_\tau) = \partial_\tau + Z(\tau;z;t)\partial_t, \tag{5.11}$$

where the function $Z(\tau;z;t)$ is a global meromorphic function on $\widetilde{\mathfrak{X}}$ satisfying the following properties:

1. *The poles of $Z(\tau;z;t)$ are contained in $\pi^{-1}_{\widetilde{\mathfrak{X}}/\mathfrak{X}}(D)$;*
2. *The quasi-periodicity:*

$$Z(\tau;z;t+m\tau+n) = Z(\tau;z;t) + m \quad \textit{for } (m,n)\in\mathbb{Z}^2. \tag{5.12}$$

Proof. Since the action of $(m,n)\in\mathbb{Z}^2$ on $\widetilde{\mathfrak{X}}$ sends ∂_τ, ∂_{z_i}, and ∂_t to $\partial_\tau + n\partial_t$, ∂_{z_i}, and ∂_t respectively, the expressions (5.11) are vector fields in $\pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D)$. □

Example 5.6. We can use the following function as $Z(\tau;z;t)$ for any $i_0\in\{1,\dots,L\}$. (cf. §3.1 of [FW]):

$$Z(\tau;z;t) = Z_{1,1}(\tau;t-z_{i_0}), \quad Z_{1,1}(\tau;t) = -\frac{1}{2\pi i}\frac{\theta'_{[0,0]}(t)}{\theta_{[0,0]}(t)}. \tag{5.13}$$

(See Appendix A for the notation of the theta functions.)

Lemma 5.7. *The connections D on $\mathcal{CC}(\mathcal{M})$ and D^* on $\mathcal{CB}(\mathcal{M})$ possess the following expressions:*

$$D_{\partial/\partial z_i} = \partial_{z_i} - \rho_i(T\{\partial_{\xi_i}\}), \tag{5.14}$$

$$D_{\partial/\partial\tau} = \partial_\tau + T\{Z(\tau;z;t)\partial_t\} = \partial_\tau + \sum_{i=1}^L \rho_i(T\{Z(\tau;z;z_i+\xi_i)\partial_{\xi_i}\}), \tag{5.15}$$

$$D^*_{\partial/\partial z_i} = \partial_{z_i} - \rho^*_i(T\{\partial_{\xi_i}\}), \tag{5.16}$$

$$D^*_{\partial/\partial\tau} = \partial_\tau + T^*\{Z(\tau;z;t)\partial_t\} = \partial_\tau + \sum_{i=1}^L \rho^*_i(T\{Z(\tau;z;z_i+\xi_i)\partial_{\xi_i}\}). \tag{5.17}$$

Proof. In the local coordinate $(\tau;z;\xi_j)$ with $\xi_j = t - z_j$, the vector fields given by (5.11) can be represented in the following forms around Q_i:

$$a_{\dot{\mathfrak{X}}}(\partial_{z_i}) = \partial_{z_i} - \delta_{i,j}\partial_{\xi_j}, \quad a_{\dot{\mathfrak{X}}}(\partial_\tau) = \partial_\tau + Z(\tau;z;z_j+\xi_j)\partial_{\xi_j}. \tag{5.18}$$

Namely, their images in $\mathcal{V}ir_S^D$ are of the following forms:

$$a_{\dot{\mathfrak{X}}}(\partial_{z_i}) = (\partial_{z_i}; (-\delta_{i,j}\partial_{\xi_j})_{j=1}^L; 0), \quad a_{\dot{\mathfrak{X}}}(\partial_\tau) = (\partial_\tau; (Z(\tau; z; z_j + \xi_j)\partial_{\xi_j})_{j=1}^L; 0), \tag{5.19}$$

whose actions on $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ can be written in the forms (5.14), (5.15), (5.16), and (5.17). □

Remark 5.8. The explicit formulae (5.16) and (5.17) of the flat connection coincide with the expressions of ∇_τ and ∇_{z_i} in §3.1 of [FW]. In order to prove the flatness of their connection, Felder and Wieczerkowski use the explicit expression of ∇_{z_i} which corresponds to (5.24) in our case. However in our construction the flatness of the connection is a priori obvious. Lemma 4.5 is the key lemma for the proof the flatness. Our proof of the flatness can be also applied to Proposition 3.4 of [FW].

5.2. **Elliptic Knizhnik-Zamolodchikov equations.** In this subsection, we show that the flat connection on the sheaf of conformal blocks defined in §5.1 is nothing but the elliptic Knizhnik-Zamolodchikov equation introduced by Etingof [E], when $\mathcal{M} = \mathcal{M}_k(V)$.

Let V_i be a finite-dimensional irreducible representation of $\mathfrak{g} = sl_N(\mathbb{C})$ for each i and put $V := \bigotimes_{i=1}^L V_i$. In view of Proposition 3.4, the connections D on $\mathcal{CC}_k(V)$ and D^* on $\mathcal{CB}_k(V)$ are regarded as connections on $V \otimes \mathcal{O}_S$ and on $V^* \otimes \mathcal{O}_S$ respectively and are dual to each other. In order to give the explicit expressions of these connections, we define the function $Z_{a,b}(t) = Z_{a,b}(\tau; t)$ by

$$Z_{a,b}(t) := \frac{w_{a,b}(t)}{4\pi i}\left(\frac{\theta'_{[a,b]}(t)}{\theta_{[a,b]}(t)} - \frac{\theta'_{[a,b]}}{\theta_{[a,b]}}\right). \tag{5.20}$$

The point $t = 0$ is an apparent singularity of this function and we can analytically continue $Z_{a,b}(t)$ to $t = 0$ by

$$Z_{a,b}(0) := \frac{1}{4\pi i}\left(\frac{\theta''_{[a,b]}}{\theta_{[a,b]}} - \left(\frac{\theta'_{[a,b]}}{\theta_{[a,b]}}\right)^2\right). \tag{5.21}$$

Theorem 5.9. *As operators acting on $V \otimes \mathcal{O}_S$ and $V^* \otimes \mathcal{O}_S$, the operators D_μ and D^*_μ for $\mu = \partial/\partial z_i, \partial/\partial\tau$ possess the following expressions:*

$$D_{\partial/\partial z_i} = \frac{\partial}{\partial z_i} - \frac{1}{\kappa}\sum_{j\neq i}\sum_{(a,b)\neq(0,0)} w_{a,b}(z_j - z_i)\rho_j(J_{a,b})\rho_i(J^{a,b}), \tag{5.22}$$

$$D_{\partial/\partial\tau} = \frac{\partial}{\partial\tau} + \frac{1}{\kappa}\sum_{i,j=1}^L\sum_{(a,b)\neq(0,0)} Z_{a,b}(z_j - z_i)\rho_j(J_{a,b})\rho_i(J^{a,b}), \tag{5.23}$$

$$D^*_{\partial/\partial z_i} = \frac{\partial}{\partial z_i} + \frac{1}{\kappa}\sum_{j\neq i}\sum_{(a,b)\neq(0,0)} w_{a,b}(z_j - z_i)\rho^*_j(J_{a,b})\rho^*_i(J^{a,b}), \tag{5.24}$$

$$D^*_{\partial/\partial\tau} = \frac{\partial}{\partial\tau} - \frac{1}{\kappa}\sum_{i,j=1}^L\sum_{(a,b)\neq(0,0)} Z_{a,b}(z_j - z_i)\rho^*_j(J_{a,b})\rho^*_i(J^{a,b}), \tag{5.25}$$

where $\rho_i(A_i)$ and $\rho^*_i(A_i)$ for $A_i \in \mathfrak{g}$ act on the i-th factor V_i of V and on the i-th factor V^*_i of V^* respectively. Here, for each $(a,b) \neq (0,0)$, the function $w_{a,b}(t) =$

$w_{a,b}(\tau;t)$ is defined by (1.14) *and the function $Z_{a,b}(t) = Z_{a,b}(\tau;t)$ by* (5.20) *and* (5.21).

Proof. Since the connections D and D^* are dual to each other, for the proof of the proposition it suffices to obtain either the formulae for D_μ (i.e., (5.22) and (5.23)) or those for D^*_μ (i.e., (5.24) and (5.25)). The formulae (5.24) and (5.25) can be proved in the same way as Lemma 2.2 or the statements in §6 of [FFR]. But we shall give the proof of (5.22) and (5.23) using the expression (4.36) of $\mathcal{T}\{\theta\}$ in Lemma 4.18.

Let us fix $v \in V^* \otimes \mathcal{O}_S \subset \mathcal{M} := \mathcal{M}_k(V)$. We rewrite $\rho_i(T\{\partial_{\xi_i}\})v$ and $T\{Z(\tau;z;t)\partial_t\}v$ in Lemma 5.7 modulo $\mathfrak{g}^D_{\mathfrak{X}}\mathcal{M}$ in terms of operators acting on V, applying the Ward identity (3.22) in the following form:

$$\rho_i(J^{a,b} \otimes w^n_{-a,-b}(\xi_i))v' \equiv -\sum_{j\neq i} w^n_{-a,-b}(z_j - z_i)\rho_j(J^{a,b})v', \tag{5.26}$$

$$\rho_i(J^{a,b} \otimes \xi^{-n-1})v' \equiv (-1)^{n+1} w_{-a,-b,n}\rho_i(J^{a,b})v' - \sum_{j\neq i} w^n_{-a,-b}(z_j - z_i)\rho_j(J^{a,b})v', \tag{5.27}$$

for $v' \in V \otimes \mathcal{O}_S$ and $n \geqq 0$. Here we used the Laurent expansion (4.28) and Lemma 4.15.

First we prove (5.22) from (5.14). The formula (4.36) for $\theta = (\delta_{i,j}\partial_{\xi_j})^L_{j=1}$ together with Lemma 4.15 shows

$$\rho_i(T\{\partial_{\xi_i}\})v \equiv -\frac{1}{2\kappa} \sum_{\substack{(a,b)\neq(0,0)\\ i'=1,\dots,L}} \rho_i(J^{a,b} \otimes w_{-a,-b}(z_i - z_{i'} + \xi_i))\rho_{i'}(J_{a,b})v. \tag{5.28}$$

Applying the Ward identity (5.26) ($n = 0$) to the terms with $i' = i$ in the right-hand side of (5.28), we find that

$$\begin{aligned}\rho_i(T\{\partial_{\xi_i}\})v \equiv\ & \frac{1}{2\kappa} \sum_{(a,b)\neq(0,0)} \sum_{j'\neq i} w_{-a,-b}(z_{j'} - z_i)\rho_{j'}(J^{a,b})\rho_i(J_{a,b})v \\ & - \frac{1}{2\kappa} \sum_{(a,b)\neq(0,0)} \sum_{i'\neq i} w_{-a,-b}(z_i - z_{i'})\rho_i(J^{a,b})\rho_{i'}(J_{a,b})v.\end{aligned} \tag{5.29}$$

Renumbering the indices of the first sum by $(a,b) \mapsto (-a,-b)$ and applying (1.17) to the second, we conclude that

$$\rho_i(T\{\partial_{\xi_i}\})v \equiv \frac{1}{\kappa} \sum_{(a,b)\neq(0,0)} \sum_{j\neq i} w_{a,b}(z_j - z_i)\rho_j(J_{a,b})\rho_i(J^{a,b})v.$$

This means that the operator $D_{\partial/\partial z_i}$ acting on $V \otimes \mathcal{O}_S$ is represented as (5.22).

The expression (5.23) of $D_{\partial/\partial\tau}$ can be deduced from (5.15) in the similar manner, which is however more involved. Let us use here the function $Z(t) = Z(\tau;z;t)$ given by Example 5.6. Then $Z(t)$ is regular at $t \neq z_{i_0}$ and the Laurent expansion of $Z(t)$ in $\xi_{i_0} = t - z_{i_0}$ is represented as

$$Z(z_{i_0} + \xi_{i_0}) = -\frac{1}{2\pi i}(\xi_{i_0}^{-1} + Z_1\xi_{i_0} + O(\xi_{i_0}^3)), \quad Z_1 := \frac{\theta'''_{[0,0]}}{3\theta'_{[0,0]}}. \tag{5.30}$$

The formula (4.36) and Lemma 4.15 imply

$$T\{Z(\tau;z;t)\partial_t\}v \equiv -\frac{1}{2\kappa} \sum_{(a,b)\neq(0,0)} \sum_{i,j=1}^{L} v_{a,b,i,j}, \tag{5.31}$$

where the vectors $v_{a,b,i,j}$ are defined by

$$v_{a,b,i,j} := \rho_j(J^{a,b} \otimes w_{-a,-b}(z_j - z_i + \xi_j)Z(z_j + \xi_j))\rho_i(J_{a,b})v.$$

Using the Laurent expansions (1.15), (5.30), and the Ward identities (5.26) ($n = 0$) and (5.27) ($n = 0, 1$), we can find the following expressions for the vectors $v_{a,b,i,j}$:

- If $j \neq i$ and $j \neq i_0$, then

$$v_{a,b,i,j} = w_{-a,-b}(z_j - z_i)Z(z_j)\rho_j(J^{a,b})\rho_i(J_{a,b})v.$$

- If $j = i$ and $j \neq i_0$, then

$$v_{a,b,i,i} \equiv Z'(z_i)\rho_i(J^{a,b})\rho_i(J_{a,b})v - \sum_{j'\neq i} w_{-a,-b}(z_{j'} - z_i)Z(z_i)\rho_{j'}(J^{a,b})\rho_i(J_{a,b})v.$$

- If $j \neq i$ and $j = i_0$, then

$$\begin{aligned} v_{a,b,i,i_0} \equiv& -\frac{1}{2\pi i}(-w_{-a,-b,0}w_{-a,-b}(z_{i_0} - z_i) + w'_{-a,-b}(z_{i_0} - z_i))\rho_{i_0}(J^{a,b})\rho_i(J_{a,b})v \\ &+ \frac{1}{2\pi i} \sum_{j'\neq i_0} w_{-a,-b}(z_{i_0} - z_i)w_{-a,-b}(z_{j'} - z_{i_0})\rho_{j'}(J^{a,b})\rho_i(J_{a,b})v. \end{aligned}$$

- If $j = i$ and $j = i_0$, then

$$\begin{aligned} v_{a,b,i_0,i_0} \equiv& -\frac{1}{2\pi i}(2w_{-a,-b,1} - (w_{-a,-b,0})^2 + Z_1)\rho_{i_0}(J^{a,b})\rho_{i_0}(J_{a,b})v \\ &+ \frac{1}{2\pi i} \sum_{j'\neq i_0} (-w'_{-a,-b}(z_{j'} - z_{i_0}) + w_{-a,-b,0}w_{-a,-b}(z_{j'} - z_{i_0}))\rho_{j'}(J^{a,b})\rho_{i_0}(J_{a,b})v. \end{aligned}$$

Note that $w^1_{-a,-b}(t) = -w'_{-a,-b}(t)$. Substituting these expressions into (5.31), we obtain an expression for $T\{Z(\tau;z;t)\partial_t\}v$ like

$$T\{Z(\tau;z;t)\partial_t\}v = -\frac{1}{2\kappa} \sum_{\substack{(a,b)\neq(0,0) \\ i,j=1,\dots,L}} Z_{a,b,i,j}\rho_j(J_{a,b})\rho_i(J^{a,b})v \quad \text{with } Z_{a,b,j,i} = Z_{-a,-b,i,j}. \tag{5.32}$$

Note that $\rho_j(J^{a,b})\rho_i(J_{a,b}) = \rho_j(J_{-a,-b})\rho_i(J^{-a,-b})$. It is possible to compute all $Z_{a,b,i,j}$ directly, but we take a short cut. Since the coefficients $Z_{a,b,i,j}$ do not depend on the representations V_i and the choice of $Z(t) = Z(\tau;z;t)$, for the determination of all $Z_{a,b,i,j}$, we have only to calculate $Z_{a,b,i_0,j}$ for $j = 1, \dots, L$. Picking up the terms which should be contained in $Z_{a,b,i_0,j}$ from the expressions for $v_{a,b,i,j}$, we obtain

$$Z_{a,b,i_0,i_0} = -\frac{1}{2\pi i}(2w_{a,b,1} - (w_{a,b,0})^2 + Z_1), \tag{5.33}$$

$$Z_{a,b,i_0,j} = w_{a,b}(z_j - z_{i_0})Z(z_j) - \frac{1}{2\pi i}(-w_{a,b,0}w_{a,b}(z_j - z_{i_0}) + w'_{a,b}(z_j - z_{i_0})), \tag{5.34}$$

where we assume $j \neq i_0$. (Here we have used the formulae (1.17) and $w_{-a,-b,\nu} = (-1)^{\nu+1} w_{a,b,\nu}$.) Note that we have

$$w'_{a,b}(t) = w_{a,b}(t)(\log w_{a,b}(t))' = w_{a,b}(t)\left(\frac{\theta'_{[a,b]}(t)}{\theta_{[a,b]}(t)} - \frac{\theta'_{[0,0]}(t)}{\theta_{[0,0]}(t)}\right). \tag{5.35}$$

Substituting (1.14), (1.16), (5.35), and the expression of Z_1 in (5.30) to (5.33) and (5.34), we can find the following results:

$$Z_{a,b,i_0,i_0} = -2Z_{a,b}(0) = -2Z_{a,b}(z_{i_0} - z_{i_0}), \tag{5.36}$$

$$Z_{a,b,i_0,j} = -2Z_{a,b}(z_j - z_{i_0}). \tag{5.37}$$

These formulae complete the proof of (5.23). □

Remark 5.10. Etingof found in [E] that certain twisted traces F of vertex operators of $\widehat{sl}_N$ satisfy $D^*_{\partial/\partial z_i} F = 0$ and $D^*_{\partial/\partial\tau} F = 0$, namely, that they are flat sections of the connection D^*.

5.3. **Modular invariance of the flat connections.** In [E], Etingof proved the modular invariance of the elliptic Knizhnik-Zamolodchikov equations by explicit computation. In this subsection, we give a more geometric proof of this fact without use of the explicit formulae of the connections. A similar property was proved for the non-twisted WZW model on elliptic curves in [FW].

For the brevity of notation, we introduce the following symbols:

- Put $\Gamma := SL_2(\mathbb{Z})$;
- In the following, we assume that the symbol γ always denote an arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$;
- For $s = (\tau; z) \in S$, $t \in \mathbb{C}$, and $i = 1, \dots, L$, put

$$\check{\tau} := \frac{a\tau + b}{c\tau + d}, \quad \check{z} = (\check{z}_i)_{i=1}^L := \frac{z}{c\tau + d} = \left(\frac{z_i}{c\tau + d}\right)_{i=1}^L,$$
$$\check{t} := \frac{t}{c\tau + d}, \quad \check{\xi}_i := \frac{\xi_i}{c\tau + d}, \quad \check{s} := (\check{\tau}; \check{z}).$$

As is well-known, $\gamma \in \Gamma$ acts on S by

$$\gamma \cdot s := \check{s} = (\check{\tau}; \check{z}) \quad \text{for } s = (\tau; z) \in S. \tag{5.38}$$

In order to define actions of Γ on $\mathfrak{X}$, $G^{\mathrm{tw}}_{\mathfrak{X}}$ and $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$, we first extend the actions of $\mathbb{Z}^2$ on $\widetilde{\mathfrak{X}}$, G, and $\mathfrak{g}$ defined by (3.1), (3.3), and (3.4) respectively to those of the semi-direct product group $\widetilde{\Gamma} := \mathbb{Z}^2 \rtimes \Gamma$ defined by

$$(m, n; \gamma)(m', n'; \gamma') := (m + m'd - n'c, n - m'b + n'a; \gamma\gamma')$$

for $(m, n; \gamma), (m', n'; \gamma') \in \mathbb{Z}^2 \times \Gamma$. Then we have

$$(m, n; 1)(0, 0; \gamma) = (0, 0; \gamma)(ma + nc, mb + nd; 1) \tag{5.39}$$

for $(m, n) \in \mathbb{Z}^2$ and $\gamma \in \Gamma$.

The action of $\widetilde{\Gamma}$ on $\widetilde{\mathfrak{X}}$ is a standard one. An element γ of Γ acts on $\widetilde{\mathfrak{X}}$ as

$$\gamma \cdot (\tau; z; t) := (\check{\tau}; \check{z}; \check{t}) \quad \text{for } (\tau; z; t) \in \widetilde{\mathfrak{X}}.$$

This action together with (3.1) induces an action of $\widetilde{\Gamma}$ on $\widetilde{\mathfrak{X}}$.

The actions of $\widetilde{\Gamma}$ on G and $\mathfrak{g}$ are defined via the action of Γ on the Heisenberg group $\mathcal{H}_N$, the central extension of $(\mathbb{Z}/N\mathbb{Z})^2$ defined by

$$\mathcal{H}_N := \mathbb{C}^\times \times (\mathbb{Z}/N\mathbb{Z})^2.$$

Here the group structure of $\mathcal{H}_N$ is given by

$$(r; m, n)(r'; m', n') := (rr'\varepsilon^{nm'}; m + m', n + n')$$

for $(r; m, n), (r'; m', n') \in \mathcal{H}_N$. The Heisenberg group $\mathcal{H}_N$ is isomorphic to the group generated by the symbols $\hat{\alpha}$, $\hat{\beta}$, and $\hat{r}$ for $r \in \mathbb{C}^\times$ with defining relations

$$\begin{aligned} &\hat{\alpha}^N = \hat{\beta}^N = 1, \quad \hat{\alpha}\hat{\beta} = \varepsilon\hat{\beta}\hat{\alpha}, \\ &\hat{r}\hat{\alpha} = \hat{\alpha}\hat{r}, \quad \hat{r}\hat{\beta} = \hat{\beta}\hat{r}, \quad \widehat{r_1 r_2} = \hat{r}_1\hat{r}_2 \quad \text{for } r, r_i \in \mathbb{C}^\times, \end{aligned} \tag{5.40}$$

where the identification of the group with $\mathcal{H}_N$ is given by

$$(r; m, n) = \hat{r}\hat{\beta}^m\hat{\alpha}^n. \tag{5.41}$$

Thus the matrices α and β given by (1.2) define a representation of $\mathcal{H}_N$ on $\mathbb{C}^N$ by

$$\hat{\alpha}v = \alpha v, \quad \hat{\beta}v = \beta v, \quad \hat{r}v = rv \quad \text{for } v \in \mathbb{C}^N \text{ and } r \in \mathbb{C}^\times.$$

Note that this representation is irreducible. For $\gamma \in \Gamma = SL_2(\mathbb{Z})$, we can define the group automorphism $(\cdot)^\gamma$ of $\mathcal{H}_N$ as follows:

- If N is odd, then we put
$$\begin{aligned} &(r; 0, 0)^\gamma := (r; 0, 0), \\ &(1; 1, 0)^\gamma := (1; a, b), \quad (1; 0, 1)^\gamma := (1; c, d) \end{aligned} \tag{5.42}$$
for $r \in \mathbb{C}^\times$.
- If N is even, then we put
$$\begin{aligned} &(r; 0, 0)^\gamma := (r; 0, 0), \\ &(1; 1, 0)^\gamma := ((\sqrt{\varepsilon})^{ab}; a, b), \quad (1; 0, 1)^\gamma := ((\sqrt{\varepsilon})^{cd}; c, d) \end{aligned} \tag{5.43}$$
for $r \in \mathbb{C}^\times$, where $\sqrt{\varepsilon} = \exp(\pi i/N)$, a primitive $(2N)$-th root of unity.

The fact that this defines an automorphism of $\mathcal{H}_N$ follows from the presentation (5.40) and (5.41). Note that this action of Γ on $\mathcal{H}_N$ induces that of Γ on $(\mathbb{Z}/N\mathbb{Z})^2$ given by $(m, n)^\gamma = (ma + nc, mb + nd)$ for $(m, n) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $\gamma \in \Gamma$ (cf. (5.39)).

Twisting the representation of $\mathcal{H}_N$ on $\mathbb{C}^N$ by $\gamma \in \Gamma$, we obtain another irreducible representation of $\mathcal{H}_N$ on $\mathbb{C}^N$:

$$\mathcal{H}_N \times \mathbb{C}^N \to \mathbb{C}^N, \quad (h, v) \mapsto h^\gamma v.$$

Since the Heisenberg group $\mathcal{H}_N$ has a unique irreducible representation, up to isomorphism, in which $\hat{r} \in \mathcal{H}_N$ for $r \in \mathbb{C}^\times$ acts as multiplication by r (the theorem of von Neumann and Stone), using the Schur lemma, we can find that there is $x_\gamma \in GL_N(\mathbb{C})$, uniquely determined up to scalar multiplications, such that

$$h x_\gamma v = x_\gamma h^\gamma v \quad \text{for } h \in \mathcal{H}_N \text{ and } \gamma \in \Gamma. \tag{5.44}$$

The mapping $\gamma \mapsto x_\gamma$ induces a group homomorphism from $\Gamma = SL_2(\mathbb{Z})$ into $PGL_N(\mathbb{C})$, which does not depend on the choices of x_γ's. In the following, we take x_γ from $G = SL_N(\mathbb{C})$, which is uniquely determined up to factor ± 1 by γ.

Example 5.11. For $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $x_\gamma \propto (\varepsilon^{-(a-1)(b-1)})_{a,b=1}^N$.

Example 5.12. When N is odd, we can choose $x_\gamma = 1$ for $\gamma \in \Gamma(N) = \{\, \gamma \in \Gamma \mid \gamma \equiv 1 \mod N \,\}$.

The desired actions of $\widetilde{\Gamma}$ on G and $\mathfrak{g}$ are now defined by

$$\begin{aligned}
(m,n;\gamma)\cdot g &:= (\beta^m\alpha^n x_\gamma) g (\beta^m\alpha^n x_\gamma)^{-1} &&\text{for } (m,n;\gamma)\in\widetilde{\Gamma} \text{ and } g\in G,\\
(m,n;\gamma)\cdot A &:= (\beta^m\alpha^n x_\gamma) A (\beta^m\alpha^n x_\gamma)^{-1} &&\text{for } (m,n;\gamma)\in\widetilde{\Gamma} \text{ and } A\in\mathfrak{g}.
\end{aligned}$$

These are extensions of the actions of $\mathbb{Z}^2$ given by (3.3) and (3.4).

The diagonal actions of $\widetilde{\Gamma}$ on $\widetilde{\mathfrak{X}}\times G$ and $\widetilde{\mathfrak{X}}\times\mathfrak{g}$ are defined by

$$\begin{aligned}
\tilde\gamma\cdot(\tilde x; g) &:= (\tilde\gamma\cdot\tilde x;\tilde\gamma\cdot g) &&\text{for } (\tilde x; g)\in\widetilde{\mathfrak{X}}\times G \text{ and } \tilde\gamma\in\widetilde{\Gamma},\\
\tilde\gamma\cdot(\tilde x; A) &:= (\tilde\gamma\cdot\tilde x;\tilde\gamma\cdot A) &&\text{for } (\tilde x; A)\in\widetilde{\mathfrak{X}}\times \mathfrak{g} \text{ and } \tilde\gamma\in\widetilde{\Gamma}.
\end{aligned}$$

Note that these actions also do not depend on the choice of x_γ.

From the actions of $\widetilde{\Gamma}$ defined above, we obtain the induced actions of Γ on $\mathfrak{X}$, $G^{\mathrm{tw}}_{\mathfrak{X}}$, and $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ defined by

$$\begin{aligned}
\gamma\cdot x &:= \pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}}(\gamma\cdot\tilde x) &&\text{for } x = \pi_{\widetilde{\mathfrak{X}}/\mathfrak{X}}(\tilde x)\in\mathfrak{X},\\
\gamma\cdot g^{\mathrm{tw}} &:= [\gamma\cdot(\tilde x; g)] &&\text{for } g^{\mathrm{tw}} = [(\tilde x; g)]\in G^{\mathrm{tw}}_{\mathfrak{X}},\\
\gamma\cdot A^{\mathrm{tw}} &:= [\gamma\cdot(\tilde x; A)] &&\text{for } A^{\mathrm{tw}} = [(\tilde x; A)]\in \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}},
\end{aligned}$$

where $\tilde x\in\widetilde{\mathfrak{X}}$, $(\tilde x; g)\in\mathfrak{X}\times G$, and $(\tilde x; A)\in\widetilde{\mathfrak{X}}\times\mathfrak{g}$ are representatives of $x\in\mathfrak{X}$, $g^{\mathrm{tw}}\in G^{\mathrm{tw}}_{\mathfrak{X}}$, and $(\tilde x; A)\in\mathfrak{X}\times\mathfrak{g}$ respectively and $\gamma\in\Gamma$ is identified with $(0,0;\gamma)\in\widetilde{\Gamma}$. (For the definitions of $\mathfrak{X}$, $G^{\mathrm{tw}}_{\mathfrak{X}}$, and $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$, see (3.2) and (3.5).) The projections $\widetilde{\mathfrak{X}}\twoheadrightarrow S$, $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}\twoheadrightarrow\mathfrak{X}$, etc. are equivariant with respect to the actions of Γ.

Moreover we obtain the following induced equivariant actions of $\gamma\in\Gamma$ on $\mathcal{T}_S$, $\mathfrak{g}^D_S$, and $\mathcal{V}ir^D_S$:

- The biholomorphic map $\gamma : S\to S$ induces the Lie algebra isomorphism $(\cdot)^\gamma : \gamma^{-1}\mathcal{T}_S \xrightarrow{\sim} \mathcal{T}_S$ of vector fields:
$$\mu^\gamma := \mu_0(\check s)\left((c\tau+d)^2\partial_\tau + \sum_{i=1}^{L} c(c\tau+d)z_i\partial_{z_i}\right) + \sum_{i=1}^{L}\mu_i(\check s)(c\tau+d)\partial_{z_i} \tag{5.45}$$
for $\mu = \mu_0(s)\partial_\tau + \sum_{i=1}^{L}\mu_i(s)\partial_{z_i}\in\gamma^{-1}\mathcal{T}_S$. (Formally μ^γ is obtained by the substitution of $\check s = (\check\tau;\check z)$ in $s = (\tau; z)$.)
- The Lie algebra isomorphism $(\cdot)^\gamma : \gamma^{-1}\hat{\mathfrak{g}}^D_S \xrightarrow{\sim} \hat{\mathfrak{g}}^D_S$ is defined by
$$\bigl((A_i(s;\xi_i))_{i=1}^{L}; f(s)\hat k\bigr)^\gamma := \bigl((x_\gamma^{-1}A_i(\check s;\check\xi_i)x_\gamma)_{i=1}^{L}; f(\check s)\hat k\bigr) \tag{5.46}$$
for $A_i(s;\xi_i)\in\gamma^{-1}(\mathfrak{g}\otimes\mathcal{O}_S((\xi_i)))$ and $f(s)\in\gamma^{-1}\mathcal{O}_S$.
- The Lie algebra isomorphism $(\cdot)^\gamma : \gamma^{-1}\mathcal{V}ir^D_S \xrightarrow{\sim} \mathcal{V}ir^D_S$ is defined by
$$\bigl(\mu; (\theta_i)_{i=1}^{L}; f(s)\hat c\bigr)^\gamma := \bigl(\mu^\gamma; (\mu_0(\check s)c(c\tau+d)\xi_i\partial_{\xi_i} + \theta_i(\check s;\check\xi_i)(c\tau+d)\partial_{\xi_i})_{i=1}^{L}; f(\check s)\hat c\bigr) \tag{5.47}$$
for $\mu = \mu_0(s)\partial_\tau + \sum_{i=1}^{L}\mu_i(s)\partial_{z_i}\in\gamma^{-1}\mathcal{T}_S$, $\theta_i = \theta_i(s;\xi_i)\partial_{\xi_i}\in\gamma^{-1}(\mathcal{O}_S((\xi_i))\partial_{\xi_i})$, and $f(s)\in\gamma^{-1}\mathcal{O}_S$.

The formula (5.47) reflects the fact that the vector field

$$\mu_0(\check{s})\partial_{\check{\tau}} + \sum_{i=1}^{L} \mu_i(\check{s})\partial_{\check{z}_i} + \theta_i(\check{s};\check{\xi}_i)\partial_{\check{\xi}_i}$$

represented in the local coordinate $(\check{\tau};\check{z};\check{\xi}_i)$ is equal to

$$\mu_0(\check{s})\left((c\tau+d)^2\partial_\tau + \sum_{i=1}^{L} c(c\tau+d)z_i\partial_{z_i} + c(c\tau+d)\xi_i\partial_{\xi_i}\right) + \sum_{i=1}^{L} \mu_i(\check{s})(c\tau+d)\partial_{z_i} + \theta_i(\check{s};\check{\xi}_i)(c\tau+d)\partial_{\xi_i}$$

represented in the local coordinate $(\tau;z;\xi_i)$. Therefore the following lemma is a direct consequence of the definitions above.

Lemma 5.13. *For $\gamma \in \Gamma$, the isomorphisms above satisfy the following:*

1. *The isomorphisms induce the following Lie algebra isomorphism:*

$$(\cdot)^\gamma : \gamma^{-1}\left(\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D\right) \xrightarrow{\sim} \mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D.$$

2. $(\gamma^{-1}\mathfrak{g}_{\dot{\mathfrak{X}}}^D)^\gamma = \mathfrak{g}_{\dot{\mathfrak{X}}}^D$.
3. $(\gamma^{-1}\mathcal{V}ir_{\dot{\mathfrak{X}}}^D)^\gamma = \mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ *and* $(\gamma^{-1}\mathcal{T}_{\dot{\mathfrak{X}}}^D)^\gamma = \mathcal{T}_{\dot{\mathfrak{X}}}^D$.

Under these preparations, we show the modular property of the sheaf of conformal coinvariants $\mathcal{CC}(\mathcal{M})$ and the sheaf of conformal blocks $\mathcal{CB}(\mathcal{M})$ coming from quotients of the Weyl modules.

For each $i = 1,\dots,L$, let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ and M_i a quotient of the Weyl module $M_k(V_i)$. Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M\otimes\mathcal{O}_S$. We denote by c_i the eigenvalue of the Casimir operator C_i acting on V_i given by (2.4) and put $\Delta_i := \kappa^{-1}c_i$. The Virasoro operator $\rho_i(T[0])$ acting on M is diagonalizable and each of its eigenvalues is of the form Δ_i+m, where m is a non-negative integer. Thus, fixing branches of the holomorphic functions $(c\tau+d)^{C_i/\kappa}$ on the upper half plane $\mathfrak{H}$, we obtain an operator $(c\tau+d)^{\rho(T[0])} = \prod_{i=1}^L (c\tau+d)^{\rho_i(T[0])}$ acting on $\mathcal{M}$, where we put $\rho(T[0]) := \sum_{i=1}^L \rho_i(T[0])$.

For $\gamma\in\Gamma$, define the isomorphism $(\cdot)^\gamma : \gamma^{-1}\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ by

$$v(s)^\gamma := x_\gamma^{-1}(c\tau+d)^{\rho(T[0])}v(\check{s}) \quad \text{for } v(s)\in\gamma^{-1}\mathcal{M}, \tag{5.48}$$

regarding x_γ^{-1} as an automorphism of $\mathcal{M}$ through the natural diagonal action of $G = SL_N(\mathbb{C})$ on M. This isomorphism (5.48) induces the isomorphism $(\cdot)^\gamma$: $\gamma^{-1}(\mathcal{M}^*) \xrightarrow{\sim} \mathcal{M}^*$ given by

$$\Phi(s)^\gamma := x_\gamma^{-1}(c\tau+d)^{\rho^*(T[0])}\Phi(\check{s}) \quad \text{for } \Phi(s)\in\gamma^{-1}(\mathcal{M}^*), \tag{5.49}$$

where we put $\rho^*(T[0]) := \sum_{i=1}^L \rho_i^*(T[0])$.

Lemma 5.14. *For $P\in\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D$, $v\in\mathcal{M}$, and $\gamma\in\Gamma$, we have*

$$(P\cdot v)^\gamma = P^\gamma \cdot v^\gamma$$

Proof. Since $\rho_i(T\{\xi_i\partial_{\xi_i}\}) = -\rho_i(T[0])$, we have the following identity of operators acting on $\mathcal{M}$ (cf. (5.47)):

$$\begin{aligned} &\mu_0(\check{s})(c\tau+d)^2\partial_\tau + \sum_{i=1}^{L} \rho_i(T\{\mu_0(\check{s})c(c\tau+d)\xi_i\partial_{\xi_i}\}) \\ &\quad = (c\tau+d)^{\rho(T[0])} \cdot (\mu_0(\check{s})(c\tau+d)^2\partial_\tau) \cdot (c\tau+d)^{-\rho(T[0])}. \end{aligned} \tag{5.50}$$

Similarly, the definition of $\mathcal{V}ir_S^D \ltimes \hat{\mathfrak{g}}_S^D$ (4.10) leads to the following identities:

$$e^{-r\rho_i(T[0])}\rho_i(A_i \otimes f_i(\xi_i))e^{r\rho_i(T[0])} = \rho_i(A_i \otimes f_i(e^r\xi_i)), \tag{5.51}$$

$$e^{-r\rho_i(T[0])}\rho_i(\theta_i(\xi_i)\partial_{\xi_i})e^{r\rho_i(T[0])} = \rho_i(\theta_i(e^r\xi_i)e^{-r}\partial_{\xi_i}), \tag{5.52}$$

$$g\,\rho_i(\theta_i(\xi_i)\partial_{\xi_i}) = \rho_i(\theta_i(\xi_i)\partial_{\xi_i})\,g \tag{5.53}$$

for $A_i \otimes f_i(\xi_i) \in \mathfrak{g} \otimes \mathcal{O}_S((\xi_i))$, $r \in \mathbb{C}$, $\theta_i(\xi_i)\partial_{\xi_i} \in \mathcal{O}((\xi_i))\partial_{\xi_i}$, and $g \in G$. Hence we have

$$\begin{aligned} &(c\tau+d)^{-\rho(T[0])}x_\gamma \cdot \big((A_i(s;\xi_i))_{i=1}^L; f(s)\hat{k}\big)^\gamma \cdot x_\gamma^{-1}(c\tau+d)^{\rho(T[0])} \\ &\quad = \big((A_i(\check{s};\xi_i))_{i=1}^L; f(\check{s})\hat{k}\big), \end{aligned} \tag{5.54}$$

for $A_i(s;\xi_i) \in \gamma^{-1}(\mathfrak{g} \otimes \mathcal{O}_S((\xi_i)))$, $f(s) \in \gamma^{-1}\mathcal{O}_S$ because of (5.46), and

$$\begin{aligned} &(c\tau+d)^{-\rho(T[0])}x_\gamma \cdot \big(\mu; (\theta_i(s;\xi_i))_{i=1}^L; g(s)\hat{c}\big)^\gamma \cdot x_\gamma^{-1}(c\tau+d)^{\rho(T[0])} \\ &\quad = \big(\mu^\gamma; (\theta_i(\check{s};\xi_i)\partial_{\xi_i})_{i=1}^L; g(\check{s})\hat{c}\big) \end{aligned} \tag{5.55}$$

for $\mu = \mu_0(s)\partial_\tau + \sum_{i=1}^L \mu_i(s)\partial_{\xi_i} \in \gamma^{-1}\mathcal{T}_S$, $\theta_i(s;\xi_i)\partial_{\xi_i} \in \gamma^{-1}(\mathcal{O}_S((\xi_i))\partial_{\xi_i})$, $g(s) \in \gamma^{-1}\mathcal{O}_S$ because of (5.47). These formulae prove the lemma. $\square$

From Lemma 5.13 and Lemma 5.14 follow the modular invariance of $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$, and the modular transformations of connections D (5.1) and D^* (5.2). The results are summarized in the following theorem.

Theorem 5.15. *For each $i = 1, \dots, L$, let V_i be a finite-dimensional irreducible representation of $\mathfrak{g}$ and M_i a quotient of the Weyl module $M_k(V_i)$. Put $M := \bigotimes_{i=1}^L M_i$ and $\mathcal{M} := M \otimes \mathcal{O}_S$. Let γ be in $\Gamma = SL_2(\mathbb{Z})$. Then the isomorphisms $(\cdot)^\gamma : \gamma^{-1}\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ and $(\cdot)^\gamma : \gamma^{-1}(\mathcal{M}^*) \xrightarrow{\sim} \mathcal{M}^*$ defined by* (5.48) *and* (5.49) *induce the isomorphisms*

$$\begin{aligned} &(\cdot)^\gamma : \gamma^{-1}\mathcal{CC}(\mathcal{M}) \xrightarrow{\sim} \mathcal{CC}(\mathcal{M}), \quad v(s) \mapsto x_\gamma^{-1}(c\tau+d)^{\rho(T[0])}v(\check{s}), \\ &(\cdot)^\gamma : \gamma^{-1}\mathcal{CB}(\mathcal{M}) \xrightarrow{\sim} \mathcal{CB}(\mathcal{M}), \quad \Phi(s) \mapsto x_\gamma^{-1}(c\tau+d)^{\rho^*(T[0])}\Phi(\check{s}), \end{aligned}$$

where x_γ is an element of G satisfying (5.44). *Furthermore these isomorphisms correspond local flat sections with respect to the connections D and D^* to local flat sections. Namely, denoting the subsheaf of local flat sections of $\mathcal{CC}(\mathcal{M})$ and $\mathcal{CB}(\mathcal{M})$ by $\mathcal{CC}(\mathcal{M})^D$ and $\mathcal{CB}(\mathcal{M})^{D^*}$ respectively, we obtain the following induced isomorphisms:*

$$\begin{aligned} &(\cdot)^\gamma : \gamma^{-1}(\mathcal{CC}(\mathcal{M})^D) \xrightarrow{\sim} \mathcal{CC}(\mathcal{M})^D, \\ &(\cdot)^\gamma : \gamma^{-1}(\mathcal{CB}(\mathcal{M})^{D^*}) \xrightarrow{\sim} \mathcal{CB}(\mathcal{M})^{D^*}. \end{aligned}$$

The modular transformations of the connections are represented as

$$\begin{aligned} D_{\mu^\gamma} &= \mu^\gamma + \sum_{i=1}^{L} \rho_i(T\{\theta_i(\check{s};\check{\xi}_i)\partial_{\check{\xi}_i}\}) - \mu_0(\check{s})\, c(c\tau+d)\rho(T[0]) \\ &= [D_\mu]_{s\mapsto\check{s}} - \mu_0(\check{s})\, c(c\tau+d)\rho(T[0]), \end{aligned} \tag{5.56}$$

$$\begin{aligned} D^*_{\mu^\gamma} &= \mu^\gamma + \sum_{i=1}^{L} \rho^*_i(T\{\theta_i(\check{s};\check{\xi}_i)\partial_{\check{\xi}_i}\}) - \mu_0(\check{s})\, c(c\tau+d)\rho^*(T[0]) \\ &= \left[D^*_\mu\right]_{s\mapsto\check{s}} - \mu_0(\check{s})\, c(c\tau+d)\rho^*(T[0]), \end{aligned} \tag{5.57}$$

where $\mu = \mu_0(s)\partial_\tau + \sum_{i=1}^L \mu_i(s)\partial_{z_i} \in \gamma^{-1}\mathcal{T}_S$*,* $\theta_i = \theta_i(s;\xi_i)\partial_{\xi_i} \in \gamma^{-1}(\mathcal{O}_S((\xi_i))\partial_{\xi_i})$*,* $(\mu;(\theta_i)_{i=1}^L;0) \in \gamma^{-1}(\mathcal{V}ir^D_{\mathfrak{X}})$*, and* $[\,\cdot\,]_{s\mapsto\check{s}}$ *denotes the substitution of* $\check{s}$ *in* s*, namely,* $[\mu + A(s)]_{s\mapsto\check{s}} = \mu^\gamma + A(\check{s})$ *for* $\mu \in \gamma^{-1}\mathcal{T}_S$ *and* $A(s) \in \gamma^{-1}\,\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{CC}(\mathcal{M}))$ *or* $A(s) \in \gamma^{-1}\,\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{CB}(\mathcal{M}))$*.*

Indeed (5.56) and (5.57) follow from the explicit expressions (5.9) and (5.10) and the definition (5.47).

Using Proposition 3.4, we can identify $\mathcal{CC}_k(V)$ and $\mathcal{CB}_k(V)$ with $V \otimes \mathcal{O}_S$ and $V^* \otimes \mathcal{O}_S$ respectively. Put $\Delta := \sum_{i=1}^L \Delta_i$. Then the theorem above implies the following corollary.

Corollary 5.16. *Then, under the identifications* $\mathcal{CC}_k(V) = V\otimes\mathcal{O}_S$ *and* $\mathcal{CB}_k(V) = V^*\otimes\mathcal{O}_S$*, the isomorphisms* $(\cdot)^\gamma : \gamma^{-1}\mathcal{CC}_k(V) \xrightarrow{\sim} \mathcal{CC}_k(V)$ *and* $(\cdot)^\gamma : \gamma^{-1}\mathcal{CB}_k(\mathcal{M}) \xrightarrow{\sim} \mathcal{CB}_k(V)$ *are of the following forms:*

$$\begin{aligned} &(\cdot)^\gamma : \gamma^{-1}(V\otimes\mathcal{O}_S) \xrightarrow{\sim} V\otimes\mathcal{O}_S, && v(s) \mapsto x_\gamma^{-1}(c\tau+d)^{\Delta} v(\check{s}), \\ &(\cdot)^\gamma : \gamma^{-1}(V^*\otimes\mathcal{O}_S) \xrightarrow{\sim} V^*\otimes\mathcal{O}_S, && F(s) \mapsto x_\gamma^{-1}(c\tau+d)^{-\Delta} F(\check{s}). \end{aligned}$$

The modular transformations of the connections expressed as in Theorem 5.9 are represented as

$$D_{\mu^\gamma} = [D_\mu]_{s\mapsto\check{s}} - \mu_0(\check{s})c(c\tau+d)\Delta, \tag{5.58}$$

$$D^*_{\mu^\gamma} = \left[D^*_\mu\right]_{s\mapsto\check{s}} + \mu_0(\check{s})c(c\tau+d)\Delta, \tag{5.59}$$

where $\mu = \mu_0(s)\partial_\tau + \sum_{i=1}^L \mu_i(s)\partial_{\xi_i} \in \gamma^{-1}\mathcal{T}_S$ *and* $[\,\cdot\,]_{s\mapsto\check{s}}$ *denotes the substitution of* $\check{s}$ *in* s*, namely,* $[\mu+A(s)]_{s\mapsto\check{s}} = \mu^\gamma + A(\check{s})$ *for* $\mu\in\gamma^{-1}\mathcal{T}_S$ *and* $A(s) \in \gamma^{-1}\,\mathcal{E}nd_{\mathcal{O}_S}(V\otimes\mathcal{O}_S)$ *or* $A(s) \in \gamma^{-1}\,\mathcal{E}nd_{\mathcal{O}_S}(V^*\otimes\mathcal{O}_S)$*.*

Example 5.17. Applying (5.59) to $\mu = \partial_{z_i}$ and $\mu = \partial_\tau$, we obtain

$$\begin{aligned} D^*_{(\partial/\partial z_i)^\gamma} &= \left[D^*_{\partial/\partial z_i}\right]_{s\mapsto\check{s}}, \\ D^*_{(\partial/\partial\tau)^\gamma} &= \left[D^*_{\partial/\partial\tau}\right]_{s\mapsto\check{s}} + c(c\tau+d)\Delta, \end{aligned}$$

which were found by Etingof [E], §4.

6. Concluding remarks

We have examined a twisted WZW model on elliptic curves which gives the XYZ Gaudin model at the critical level and Etingof's elliptic KZ equations at the off-critical level. We make several comments on the related interesting problems to be solved.

Factorization. We have studied the twisted WZW model only on a family of smooth pointed elliptic curves, but as in [TUY] we can also consider the model on a family of stable pointed elliptic curves. By extending the connections acting on the sheaves of conformal blocks to those with regular singularities at the boundary of the family, we shall be able to establish the equivalence between our geometric approach and Etingof's approach by means of twisted traces of the products of twisted vertex operators on the Riemann sphere. Furthermore we shall be able to obtain a dimension formulae for the spaces of conformal blocks. A detailed investigation shall be given in a forthcoming paper.

Generalization to higher genus. Bernard generalized the KZB equations to higher genus Riemann surfaces in [Be2]. In [Fe], Felder established the geometric interpretation of the KZB equations on Riemann surfaces by the non-twisted WZW models clarifying the notion of the dynamical r-matrices in higher genus cases. Our formulation for the twisted WZW model is also valid for arbitrary Riemann surfaces. See Appendix C for details.

Discretization. Felder calls his interpretation of the KZB equations in [Fe] "the first step of the 'St. Petersburg q-deformation recipe' in higher genus cases". We hope that our twisted WZW model on elliptic curves can also be q-deformed. The resulting "elliptic q-KZ equations", for example, would be related to the difference equations proposed in [Tak2].

Intertwining vectors. Note that the Boltzmann weights of the $A^{(1)}_{N-1}$ face model can be expressed by the elliptic quantum R-matrix and the intertwining vectors ([Ba], [JMO], [DJKMO]). Therefore it can be expected that there exists a quasi-classical limit of the intertwining vectors by means of which the relation of the non-twisted and twisted WZW models on elliptic curves will be clarified.

In addition, the intertwining vectors play an important role in constructing Bethe vectors of the XYZ spin chain models ([Ba], [Tak1]) and the integral solution of the difference equations in [Tak2]. They are introduced as a kind of technical tools there, but our approach from the twisted WZW model, i.e., from the classical limit should reveal their algebro-geometric meaning.

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Appendix A. Theta functions with characteristics

Here we collect properties of theta functions of one variable used in this paper. Following [M], we use the notation:

$$\theta_{\kappa,\kappa'}(t;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\kappa)^2\tau + 2\pi i(n+\kappa)(t+\kappa')}. \tag{A.1}$$

for the theta functions with characteristics. Here t is a complex number, τ belongs to the upper half plane $\mathfrak{H}$ and κ, κ' are rational numbers. They are related with each other by shifts of t:

$$\theta_{\kappa_1+\kappa_2,\kappa'_1+\kappa'_2}(t;\tau) = e^{\pi i\kappa_2^2\tau+2\pi i\kappa_2(t+\kappa'_1+\kappa'_2)}\theta_{\kappa_1,\kappa'_1}(t+\kappa_2\tau+\kappa'_2;\tau). \tag{A.2}$$

Since the zero set of $\theta_{00}(t;\tau)$ is $\{1/2+\tau/2\}+\mathbb{Z}+\mathbb{Z}\tau$, the zero set of $\theta_{\kappa,\kappa'}(t;\tau)$ is

$$\left\{\frac{1}{2}-\kappa+\left(\frac{1}{2}-\kappa'\right)\tau\right\}+\mathbb{Z}+\mathbb{Z}\tau, \tag{A.3}$$

because of (A.2). Fundamental properties of the function $\theta_{\kappa,\kappa'}$ are the quasi-periodicity with respect to t:

$$\begin{aligned}\theta_{\kappa,\kappa'}(t+1;\tau) &= e^{2\pi i\kappa}\theta_{\kappa,\kappa'}(t;\tau),\\ \theta_{\kappa,\kappa'}(t+\tau;\tau) &= e^{-\pi i\tau-2\pi i(t+\kappa')}\theta_{\kappa,\kappa'}(t;\tau),\end{aligned} \tag{A.4}$$

and the automorphic property:

$$\begin{aligned}\theta_{\kappa,\kappa'}(t;\tau+1) &= e^{-\pi i\kappa(\kappa+1)}\theta_{\kappa,\kappa+\kappa'+\frac{1}{2}}(t;\tau),\\ \theta_{\kappa,\kappa'}\left(\frac{t}{\tau};-\frac{1}{\tau}\right) &= (-i\tau)^{1/2}e^{2\pi i\kappa\kappa'}e^{\pi it^2/\tau}\theta_{\kappa',-\kappa}(t;\tau).\end{aligned} \tag{A.5}$$

The formulae

$$\theta_{-\kappa,-\kappa'}(t;\tau) = \theta_{\kappa,\kappa'}(-t;\tau), \tag{A.6}$$

$$\theta_{\kappa+m,\kappa'+n}(t;\tau) = e^{2\pi i\kappa n}\theta_{\kappa,\kappa'}(t;\tau), \tag{A.7}$$

are easily deduced from the definition (A.1), where m and n are integers.

We use mostly the following special characteristics. Let $N \geqq 2$ be a natural number and a, b arbitrary integers. We denote

$$\theta_{[a,b]}(t;\tau) := \theta_{\frac{a}{N}-\frac{1}{2},-\frac{b}{N}+\frac{1}{2}}(t;\tau). \tag{A.8}$$

The standard abbreviations,

$$\theta_{[a,b]} := \theta_{[a,b]}(0;\tau), \quad \theta'_{[a,b]} := \left.\frac{d}{dt}\theta_{[a,b]}(t;\tau)\right|_{t=0},$$

and $\theta''_{[a,b]}$ etc. likewise defined are also used.

Lemma A.1.

$$\frac{N^2-1}{6}\frac{\theta'''_{[0,0]}}{\theta'_{[0,0]}} - \frac{1}{2}\sum_{(a,b)\neq(0,0)}\frac{\theta''_{[a,b]}}{\theta_{[a,b]}} = 0, \tag{A.9}$$

where the indices in the second term run through $a = 0,\dots,N-1$, $b = 0,\dots,N-1$ and $(a,b)\neq(0,0)$.

This is a generalization of the well-known formula

$$\frac{\theta'''_{1/2,1/2}}{\theta'_{1/2,1/2}} = \frac{\theta''_{1/2,0}}{\theta_{1/2,0}} + \frac{\theta''_{0,0}}{\theta_{0,0}} + \frac{\theta''_{0,1/2}}{\theta_{0,1/2}}, \tag{A.10}$$

which is the case $N = 2$ of Lemma A.1.

Proof. It is easy to show that

$$N \prod_{a=0}^{N-1} \prod_{b=0}^{N-1} \theta_{[a,b]}(t;\tau) = \left(\prod_{(a,b)\neq(0,0)} \theta_{[a,b]} \right) \theta_{[0,0]}(Nt;\tau). \tag{A.11}$$

In fact we have only to compare the periodicity and zeros of the both sides by using (A.4) and (A.3). The over-all coefficient can be determined by the first term (namely the coefficient of t) in the Taylor expansion around $t = 0$.

The coefficients of t^2 of the Taylor expansion of (A.11) give

$$\sum_{(a,b)\neq(0,0)} \frac{\theta'_{[a,b]}}{\theta_{[a,b]}} = 0,$$

and using this equality, we can rewrite the terms of order t^3 in (A.11) as follows:

$$\frac{N^2-1}{6} \frac{\theta'''_{[0,0]}}{\theta'_{[0,0]}} - \frac{1}{2} \sum_{(a,b)\neq(0,0)} \frac{\theta''_{[a,b]}}{\theta_{[a,b]}} = - \sum_{(a,b)\neq(0,0)} \left(\frac{\theta'_{[a,b]}}{\theta_{[a,b]}} \right)^2 . \tag{A.12}$$

Therefore, in order to prove Lemma A.1, we have to show that the right-hand side of (A.12) is zero. Let us denote it by $f(\tau)$ as a function of τ. It has following properties:

- $f(\tau)$ is a holomorphic function on the upper half plane $\mathfrak{H}$. ((A.3))
- $f(\tau)$ is bounded when $\operatorname{Im}\tau \to +\infty$. ((A.1))
- $f(\tau+1) = f(\tau)$, $f(-1/\tau) = \tau^2 f(\tau)$. ((A.5), (A.7))

Hence $f(\tau)$ is an integral modular form of weight 2 and level 1, which is nothing but zero. (See, for example, Théorèm 4 (i), §3 Chapitre VII of [Se1], Proposition 2.26 of [Sh] or Theorem 14 in Chapter II of [Sc].) This proves the lemma. □

Appendix B. The Kodaira-Spencer map of a family of Riemann surfaces

Let $\pi : \mathfrak{X} \to S$ be a family of compact Riemann surfaces over a complex manifold S and $q_i : S \to \mathfrak{X}$ a holomorphic section of π for each $i = 1, \dots, L$. Put $Q_i := q_i(S)$ and assume that $Q_i \cap Q_j = \emptyset$ if $i \neq j$. Then $D := \bigcup_{i=1}^{L} Q_i$ is a divisor of $\mathfrak{X}$ étale over S. We call $(\pi : \mathfrak{X} \to S; q_1, \dots, q_L)$ a family of pointed compact Riemann surfaces. Denote by $\mathcal{T}_{\mathfrak{X}}(-\log D)$ the sheaf of vector fields tangent to D. As in §4.1, let $\mathcal{T}_{\mathfrak{X},\pi}(-\log D)$ be the inverse image of $\pi^{-1}\mathcal{T}_S \subset \pi^*\mathcal{T}_S$ in $\mathcal{T}_{\mathfrak{X}}(-\log D)$. Then we obtain the following short exact sequence:

$$0 \to \mathcal{T}_{\mathfrak{X}/S}(-D) \to \mathcal{T}_{\mathfrak{X},\pi}(-\log D) \to \pi^{-1}\mathcal{T}_S \to 0,$$

which is a Lie algebra extension. The derived direct image of this sequence produces the following long exact sequence:

$$\cdots \to \pi_*\mathcal{T}_{\mathfrak{X},\pi}(-\log D) \to \mathcal{T}_S \to R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}(-D) \to R^1\pi_*\mathcal{T}_{\mathfrak{X},\pi}(-\log D) \to \cdots .$$

The connecting homomorphism $\mathcal{T}_S \to R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}(-D)$ is called the *Kodaira-Spencer map* of the family $(\pi : \mathfrak{X} \to S; q_1, \dots, q_L)$.

For an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ and a closed analytic subspace Z of $\mathfrak{X}$, denote by $\mathcal{F}_Z^\wedge$ the completion of $\mathcal{F}$ at Z. Consider the following exact sequences:

$$\begin{aligned}
&0 \to (\mathcal{T}_{\mathfrak{X}/S}(-D))_D^\wedge \to (\mathcal{T}_{\mathfrak{X}}(-\log D))_D^\wedge \to (\pi^*\mathcal{T}_S)_D^\wedge \to 0,\\
&0 \to (\mathcal{T}_{\mathfrak{X}/S}(*D))_D^\wedge \to (\mathcal{T}_{\mathfrak{X}}(*D))_D^\wedge \to (\pi^*\mathcal{T}_S)_D^\wedge \to 0.
\end{aligned}$$

As above, the inverse images of $\pi^{-1}\mathcal{T}_S|_D \subset (\pi^*\mathcal{T}_S)_D^\wedge$ in $(\mathcal{T}_{\mathfrak{X}}(-\log D))_D^\wedge$ and $(\mathcal{T}_{\mathfrak{X}}(*D))_D^\wedge$ is denoted by $\mathcal{T}_\pi(-\log D)$ and $\mathcal{T}_\pi(*D)$ respectively. Then we obtain the Lie algebra extensions below:

$$\begin{aligned}
&0 \to (\mathcal{T}_{\mathfrak{X}/S}(-D))_D^\wedge \to \mathcal{T}_\pi(-\log D) \to \pi^{-1}\mathcal{T}_S|_D \to 0,\\
&0 \to (\mathcal{T}_{\mathfrak{X}/S}(*D))_D^\wedge \to \mathcal{T}_\pi(*D) \to \pi^{-1}\mathcal{T}_S|_D \to 0.
\end{aligned}$$

The direct images of these exact sequences to S are also exact.

Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & (\sharp) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \pi_*\mathcal{T}_{\mathfrak{X}/S}^{-D} & \longrightarrow & T_{\dot{\mathfrak{X}}}^{*D} \oplus T^{-D} & \longrightarrow & T^{*D} & \longrightarrow & R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}^{-D} \longrightarrow 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*\mathcal{T}_{\mathfrak{X},\pi}^{-\log D} & \longrightarrow & T_{\pi,\dot{\mathfrak{X}}}^{*D} \oplus T_\pi^{-\log D} & \longrightarrow & T_\pi^{*D} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_S & \longrightarrow & \mathcal{T}_S \oplus (\mathcal{T}_S)^L & \longrightarrow & (\mathcal{T}_S)^L & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (\sharp) & & 0 & & 0 & &
\end{array} \tag{B.1}$$

where we put

$$\begin{aligned}
&\mathcal{T}_{\mathfrak{X}/S}^{-D} := \mathcal{T}_{\mathfrak{X}/S}(-D), \quad \mathcal{T}_{\mathfrak{X},\pi}^{-\log D} := \mathcal{T}_{\mathfrak{X},\pi}(-\log D),\\
&T_{\dot{\mathfrak{X}}}^{*D} := \pi_*\mathcal{T}_{\mathfrak{X}/S}(*D), \quad T^{-D} := \pi_*(\mathcal{T}_{\mathfrak{X}/S}(-D))_D^\wedge, \quad T^{*D} := \pi_*(\mathcal{T}_{\mathfrak{X}/S}(*D))_D^\wedge,\\
&T_{\pi,\dot{\mathfrak{X}}}^{*D} := \pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D), \quad T_\pi^{-\log D} := \pi_*\mathcal{T}_\pi(-\log D), \quad T_\pi^* := \pi_*\mathcal{T}_\pi(*D).
\end{aligned}$$

The horizontal and vertical sequences in the diagram (B.1) are all exact and the arrow from $\mathcal{T}_S$ to $R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}^{-D} = R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}(-D)$ through $(\sharp)$ is nothing but the Kodaira-Spencer map, which is described as follows. For $\mu \in \mathcal{T}_S$, chasing the diagram above, we can choose $(a_{\dot{\mathfrak{X}}}, a_+)$, α and $[\alpha]$ in order:

1. $(a_{\dot{\mathfrak{X}}}, a_+) \in T_{\pi,\dot{\mathfrak{X}}}^{*D} \oplus T_\pi^{-\log D}$, whose image in $\mathcal{T}_S \oplus (\mathcal{T}_S)^L$ is equal to $(\mu; (\mu)_{i=1}^L) \in \mathcal{T}_S \oplus (\mathcal{T}_S)^L$;
2. $\alpha \in T^{*D}$, whose image in T_π^{*D} is equal to $a_{\dot{\mathfrak{X}}} - a_+ \in T_\pi^{*D}$;
3. $[\alpha] \in R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}^{-D}$, which is the image of α in $R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}^{-D}$.

Then the cohomology class $[\alpha] \in R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}(-D)$ does not depend on the choice of $(a_{\dot{\mathfrak{X}}}, a_+)$ and α. The Kodaira-Spencer map sends $\mu \in \mathcal{T}_S$ to $[\alpha] \in R^1\pi_*\mathcal{T}_{\mathfrak{X}/S}(-D)$. Recall that $T_{\dot{\mathfrak{X}}}^{*D}$ and $T_{\pi,\dot{\mathfrak{X}}}^{*D}$ are denoted by $\mathcal{T}_{\dot{\mathfrak{X}}}^D$ and $\mathcal{V}ir_{\dot{\mathfrak{X}}}^D$ respectively in §5. The

short exact sequence (4.3) is included in the second vertical exact sequence in (B.1) and the lifting from $\mathcal{T}_S$ to $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ (cf. §5.1) essentially corresponds to the operation 1 above. Hence we can see that the constructions in §5 originate in the above description of the Kodaira-Spencer map.

However we considered not only T^{*D}_π but also its extension $\mathcal{V}ir^S_D$ by $\mathcal{O}_S\hat{c}$. This is a difference between the description of the Kodaira-Spencer map and the constructions in §5. We remark that Beilinson and Schechtman give the intrinsic (i.e., coordinate-free) description of the Virasoro algebras in [BS].

Appendix C. On a formulation for higher genus Riemann surfaces

In this appendix, we shall comment on a formulation of twisted Wess-Zumino-Witten models on higher genus Riemann surfaces.

Let $\pi : \mathfrak{X} \to S$, $q_i : S \to \mathfrak{X}$, $Q_i = q_i(S)$, and $D = \sum Q_i$ be the same as Appendix B. Suppose that, for each i, we can take a holomorphic function ξ_i on an open neighborhood U_i of Q_i with the property that the mapping $U_i \to S \times \xi_i(U_i)$ given by $x \mapsto (\pi(x), \xi_i(x))$ is biholomorphic and $\xi_i(Q_i) = \{0\}$.

Then, in precisely the same way as §5, we can define $\mathcal{T}_{\mathfrak{X},\pi}(*D)$, $\mathcal{T}_{\mathfrak{X}/S}(*D)$, $\mathcal{V}ir^D_S$, $\mathcal{T}^D_{\dot{\mathfrak{X}}}$, $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$, etc. We define the action of $\pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D)$ on $\mathcal{V}ir^D_S$ by

$$a_{\dot{\mathfrak{X}}} \cdot \alpha := [a_{\dot{\mathfrak{X}}}, \alpha] \quad \text{for } a_{\dot{\mathfrak{X}}} \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}},\ \alpha \in \mathcal{V}ir^D_S, \tag{C.1}$$

where $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ is identified with its image in $\mathcal{V}ir^D_S$ and the bracket of the right-hand side is the Lie algebra structure of $\mathcal{V}ir^D_S$.

We remark that the embedding $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ and that of $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ into $\mathcal{V}ir^D_S$ are not always Lie algebra homomorphisms and the action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{V}ir^D_S$ does not always preserve $\mathcal{T}^D_{\dot{\mathfrak{X}}}$. Thus we must add a supplementary structure on the Riemann surface.

Suppose that we can take an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of $\mathfrak{X}$ and a family $\{\xi_\lambda : U_\lambda \to \mathbb{C}\}_{\lambda\in\Lambda}$ of holomorphic functions satisfying the following properties:

1. For each $\lambda \in \Lambda$, the mapping $U_\lambda \to S \times \mathbb{C}$ given by $x \mapsto (\pi(x), \xi_\lambda(x))$ is a biholomorphic mapping from U_λ onto an open subset of $S \times \mathbb{C}$.
2. For any $\lambda, \lambda' \in \Lambda$ with $U_\lambda \cap U'_\lambda \neq \emptyset$, there exists $a, b, c, d \in \mathcal{O}_S(S)$ with the property that $\xi_{\lambda'} = (a\xi_\lambda + b)/(c\xi_\lambda + d)$ on $U_\lambda \cap U'_\lambda$.

We call $\{\xi_\lambda\}_{\lambda\in\Lambda}$ a *projective structure* on the family $\pi : \mathfrak{X} \to S$ of Riemann surfaces. Moreover assume that $\{\xi_\lambda\}_{\lambda\in\Lambda} \cup \{\xi_i\}_{i=1}^L$ is also a projective structure on the family.

Lemma C.1. *Under the assumption above, the action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{V}ir^D_S$ preserves $\mathcal{T}^D_{\dot{\mathfrak{X}}}$ and in particular the embedding $\mathcal{T}^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$ is a Lie algebra homomorphism.*

Proof. It suffices to show that $\mathrm{c_V}(\theta, \eta) = 0$ for $(\mu; \theta; 0) \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ and $(0; \nu; 0) \in \mathcal{T}^D_{\dot{\mathfrak{X}}}$. For this purpose, as in the proof of Lemma 4.5, it is enough to show that, for $(\mu; \theta; 0) \in \mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ and $(0; \nu; 0) \in \mathcal{T}^D_{\dot{\mathfrak{X}}}$, we can take $\omega \in \pi_*\Omega^1_{\mathfrak{X}/S}(*D)$ with the property that $\omega = \theta'''(\xi_i)\eta(\xi_i)\,d\xi_i$ near Q_i. To do this, we calculate the transformation property of $\theta'''(\xi_\lambda)\eta(\xi_\lambda)\,d\xi_\lambda$ under coordinate changes. Take any ξ, ζ from $\{\xi_\lambda\}_{\lambda\in\Lambda} \cup \{\xi_i\}_{i=1}^L$. Then there is $a, b, c, d \in \mathcal{O}_S(S)$ with $\zeta = (a\xi + b)/(c\xi + d)$. Fix a local coordinate $s = (s_i)_{i=1}^M$ on S. Then we obtain two local coordinates $(s; \xi)$ and $(s; \zeta)$ on $\mathfrak{X}$. Let $a_{\dot{\mathfrak{X}}}$ be in $\pi_*\mathcal{T}_{\mathfrak{X},\pi}(*D)$ and α in $\pi_*\mathcal{T}_{\mathfrak{X}/S}(*D)$. Then we can represent

$a_{\dot{\mathfrak{X}}}$ and α in the two local coordinates $(s;\xi)$ and $(s;\zeta)$:

$$a_{\dot{\mathfrak{X}}} = \begin{cases} \mu + \theta^{\xi}(s;\xi)\partial_{\xi} & \text{in } (s;\xi), \\ \mu + \theta^{\zeta}(s;\zeta)\partial_{\zeta} & \text{in } (s;\zeta), \end{cases} \qquad \alpha = \begin{cases} \eta^{\xi}(s;\xi)\partial_{\xi} & \text{in } (s;\xi), \\ \eta^{\zeta}(s;\zeta)\partial_{\zeta} & \text{in } (s;\zeta), \end{cases}$$

where $\mu = \sum_{i=1}^{M} \mu_i(s)\partial_{s_i}$. Then a straightforward calculation shows that

$$\frac{\partial^3\theta^{\xi}(s;\xi)}{\partial\xi^3}\eta^{\xi}(s;\xi)\,d\xi = \frac{\partial^3\theta^{\zeta}(s;\zeta)}{\partial\zeta^3}\eta^{\zeta}(s;\zeta)\,d\zeta.$$

Hence there is a unique $\omega \in \pi_*\Omega^1_{\mathfrak{X}/S}(*D)$ such that the representation of ω under the coordinate $(s;\xi)$ is equal to $(\partial_{\xi}^3\theta^{\xi}(s;\xi))\eta^{\xi}(s;\xi)\,d\xi$ for any $\xi \in \{\xi_\lambda\}_{\lambda\in\Lambda} \cup \{\xi_i\}_{i=1}^{L}$. Thus we have completed the proof. □

Example C.2. Assume that $\pi : \mathfrak{X} \to S$ denotes the family of elliptic curves defined in §3.1. Then the local coordinate t along the fibers gives a projective structure on the family. For each holomorphic section $q : S \to \mathfrak{X}$ of $\pi : \mathfrak{X} \twoheadrightarrow S$, put $\xi_q := t - q$, which is regarded as a holomorphic function on a sufficiently small open neighborhood of $q(S)$. Then the family $\{\xi_q\}$ is a projective structure on the family and contains $\{\xi_i\}_{i=1}^{L} = \{\xi_{q_i}\}_{i=1}^{L}$.

Example C.3. A family of compact Riemann surfaces given by the Schottky parametrization possess a natural projective structure, because each compact Riemann surface in that family is represented as the quotient space of the punctured Riemann sphere by the action of a Schottky group, which is generated by a certain finite set consisting of fractional linear transformations. The Schottky parametrization is used in [Be2].

We can generalize the statement of Lemma 4.7 for the family of Riemann surfaces with projective structures. Let $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ denote an $\mathcal{O}_{\mathfrak{X}}$-Lie algebra which is locally $\mathcal{O}_{\mathfrak{X}}$-free of finite rank with holomorphic flat connection ∇ and suppose that the action of a vector field on $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ via the connection is a Lie algebra derivation:

$$\nabla[A,B] = [\nabla A, B] + [A, \nabla B] \quad \text{for } A, B \in \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}.$$

Assume that the fibers of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ are (non-canonically) isomorphic to a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. The $\mathcal{O}_{\mathfrak{X}}$-inner product $(.|.)$ is defined by

$$(A|B) := \frac{1}{2h^{\vee}}\operatorname{tr}_{\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}}(\operatorname{ad} A \operatorname{ad} B) \quad \text{for } A, B \in \mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}},$$

where $h^{\vee}$ denotes the dual Coxeter number of $\mathfrak{g}$. Then the inner product is non-degenerate and invariant under the translations along ∇. We can take a local trivialization $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}} \cong \mathfrak{g} \otimes \mathcal{O}_{\mathfrak{X}}$, under which the connection is represented as the exterior derivative on $\mathfrak{X}$ (i.e., the trivial connection). We assume that we can take such a trivialization of $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ on some neighborhood of the divisor D. Under this situation, the constructions in §3 goes through in precisely the same way and then Lemma 4.7 also holds.

However, Lemma 4.10 does not always hold. The action of $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$ on $\mathcal{M}$ is not a representation but a projective representation in general, because the embedding $\mathcal{V}ir^D_{\dot{\mathfrak{X}}} \hookrightarrow \mathcal{V}ir^D_S$ is not always a Lie algebra homomorphism but so is the composition of the embedding and the natural projection $\mathcal{V}ir^D_S \twoheadrightarrow \mathcal{V}ir^D_S/\mathcal{O}\hat{c}$. Nevertheless Lemma 4.11 and Lemma 4.12 also hold in our case. Namely, the Lie algebra $\mathcal{V}ir^D_{\dot{\mathfrak{X}}}$

acts on both the sheaf $\mathcal{CC}(\mathcal{M})$ of conformal coinvariants and the sheaf $\mathcal{CB}(\mathcal{M})$ of conformal blocks. Furthermore Lemma 4.13 can be proved in the same way. Therefore we conclude that $\mathcal{CB}(\mathcal{M})$ possess a projectively flat connection. For the non-twisted Wess-Zumino-Witten models, the coordinate-free version of Lemma 4.18 for $\theta \in \mathcal{T}^D_{\dot{\mathfrak{X}}}$ is used in the proof of the main theorem 4.2 in [Ts]. Since the analogue of Lemma 4.18 for $\theta \in \mathcal{T}^D_{\dot{\mathfrak{X}}}$ can be also proved in our situation, we can obtain the projectively flat connections on $\mathcal{CC}(\mathcal{M})$.

We can generalize the setting above in various ways:

1. We can replace the family of pointed compact Riemann surfaces by that of stable pointed curves in the course of [TUY], [U], and [Ts]. Then we shall be able to show the factorization property of conformal blocks under appropriate assumptions.
2. We can consider not only deformations of pointed Riemann surfaces but also deformations of $G^{\mathrm{tw}}_{\mathfrak{X}}$-torsors (or principal bundles). For example, the Knizhnik-Zamolodchikov-Bernard equations on Riemann surfaces (cf. [Be1], [Be2], [FW], [Fe]) can be formulated on a family of pairs of compact Riemann surfaces and principal G-bundles on them, where G is a finite-dimensional simple algebraic group over $\mathbb{C}$.
3. Furthermore we can also consider Borel subgroup bundles which are subbundles of the restriction of $G^{\mathrm{tw}}_{\mathfrak{X}}$ on D and their deformations. (More generally, we can consider a family of quasi parabolic structures on G^{tw}-torsors.) Then we can define the notion of highest weight representations of the sheaf of affine Lie algebras with respect to the Borel subgroup bundles. Anyway a choice of a Borel subgroup is required by the definition of the category $\mathcal{O}$ of representations, which contains the Verma modules, their irreducible quotients, and especially the Wakimoto modules. Note that the constructions of the Wakimoto modules (cf. [FF1], [FF2], and [K]) essentially depend on the choice of a triangular decomposition (equivalently that of a Borel subalgebra) of a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$.
4. We can replace the holomorphic flat connection on $\mathfrak{g}^{\mathrm{tw}}_{\mathfrak{X}}$ by a meromorphic flat connection with regular singularity along the divisor D. Assume that the local monodromy group of the connection around D is finite. Then we can construct a sheaf of twisted affine Lie algebras at D and can define the notion of conformal blocks for representations of the twisted affine Lie algebras.

Detailed expositions shall be given in forthcoming papers.

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